<u>PSC Math 171 Essay Problems</u>

Limits and Continuity

1. State our informal definition of the limit concept.

   Suppose f is defined on an open interval containing the number c, but f need not be defined at c. If $f(x)$ can be made arbitrarily close to the number L by choosing x sufficiently close to, but different from, c then we say $\lim_{x\to c} f(x) = L$.

2. State four ways in which a limit may fail to exist. Give an example of each.

   - The limit from the left does not equal the limit from the right, e.g. $\lim_{x\to 0} |x|/x$ DNE.
   - The function values grow without bound as the limit point is approached, e.g. $\lim_{x\to 0} 1/x^2 = +\infty$.
   - The function values continually oscillate between different, fixed values, e.g. $\lim_{x\to 0} \sin(\frac{1}{x})$ DNE.
   - The function is not defined in an open interval containing the limit point, e.g. $\lim_{x\to -1} \sqrt{x}$ DNE.

3. Give an example of a function that has a limit at $x = 1$, but that limit cannot be found by direct substitution. Explain why the substitution technique does not work.

   Consider $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1}$. Direct substitution yields the indeterminate form 0/0. Substitution cannot be applied because the function is not defined at the limit point. However,

   $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = 2.$$

4. You attempted to evaluate a limit by substitution, but you came up with 0/0. State several ways that you might be able to resolve the indeterminate form 0/0. (Also be able to give an example of each.)

   - Factor and cancel
   - Expand and simplify
   - Multiply by the conjugate and simplify
   - Get a common denominator and add (or subtract)
   - Use $\lim_{x\to 0} \sin x/x = 1$
   - Use the definition of absolute value

5. What does it mean for a function to be continuous at a point?

   f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

6. Using the definition of continuity, describe three reasons why a function may fail to be continuous at a point.

   - The function is not defined at the point.
   - The limit does not exist at the point.
   - The function value does not equal the limit.

7. What is the difference between a removable discontinuity and a nonremovable discontinuity?

   The limit exists at a removable discontinuity, while the limit does not exist at a nonremovable discontinuity.

8. Suppose f is a continuous function on $[a, b]$. What does the Intermediate Value theorem tell us about the values of f on this interval?

   According the the Intermediate Value theorem, f must attain every value between $f(a)$ and $f(b)$ on the interval $[a, b]$.

Derivatives and Differentiation

1. State the formal limit definition of the derivative.

   The derivative of f at x is given by

   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

   provided this limit exists.

2. Describe three reasons why a derivative may fail to exist at a point.

   - The function is not continuous at the point.
   - The slope of the tangent line from the left is not equal to the slope of the tangent line from the right (i.e. the graph has a sharp point).
   - The tangent line at the point is vertical.

3. State three interpretations of the derivative.

   - $f'(x)$ is the slope of the line tangent to the graph of f at x.
   - $\frac{dy}{dx}$ is the rate of change of y with respect to x.
   - $f'(x)$ is a new function obtained from f by the process of differentiation.

4. State two forms of the chain rule.

   - $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$
   - If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Applications of Derivatives

1. What is a critical number?

   A critical number of f is an interior point in the domain of f at which f' is zero or f' is not defined.

2. List the steps of the process we use to find the absolute extrema of a continuous function on a closed and bounded interval.

   - Find the function's critical numbers.
   - Find the interval endpoints.
   - Evaluate the function at critical numbers and endpoints.
   - The greatest and least values are the absolute extrema.

3. What do the signs of f' say about f?

   - If f' is positive on (a, b), then f is increasing on (a, b).
   - If f' is negative on (a, b), then f is decreasing on (a, b).

4. What is an inflection point?

   An inflection point is a point at which a graph has a tangent line and the graph changes concavity.

5. What do the signs of f'' say about the graph of f?

   - If f'' is positive on (a, b), then the graph of f is concave up on (a, b).
   - If f'' is negative on (a, b), then the graph of f is concave down on (a, b).