Lecture 15: The chain rule

Objectives:

(15.1) Identify compositions of functions and write a function as a composition of two functions.

(15.2) Use the chain rule to differentiate compositions of functions.

Chain Rule

The composition of the functions f and g is the new function $f \circ g$ that satisfies $(f \circ g)(x) = f(g(x))$. The chain rule will tell us how to differentiate a composition of two functions. Before we can use the chain rule, however, we must be able to identify compositions of functions and determine what functions are being composed.

Example 1 Identify two functions f and g so that $h(x) = f(g(x))$.

1. $h(x) = (x^2 + 8)^5$

   $x^2 + 8$ is "plugged into" the 5th power function. We let $f(x) = x^5$ be the outside function and $g(x) = x^2 + 8$ be the inside function. Then we have

   $$f(g(x)) = [g(x)]^5 = (x^2 + 8)^5.$$

2. $h(x) = \sin(\sqrt{x})$

   $\sqrt{x}$ is "plugged into" the sine function. We let $f(x) = \sin x$ be the outside function and $g(x) = \sqrt{x}$ be the inside function. Then we have

   $$f(g(x)) = \sin(g(x)) = \sin(\sqrt{x}).$$

There are always infinitely many ways to write a function as a composition of two functions. When using the chain rule, we will always make the most obvious choices (as we did above).

Chain Rule

If $y = f(s)$ is a differentiable function of s and $s = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{ds} \cdot \frac{ds}{dx}$$

or

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Both forms of the chain rule given above are very useful, but they are usually useful in different applications. For routine differentiation of functions, we will normally use the second form. It basically says that to differentiate a composition, we take the derivative of the outside function, evaluate at the inside function, and then multiply by the derivative of the inside function.

Example 2 Use the chain rule to differentiate $y = (x^2 + 8)^5$.

The outside function is $f(x) = x^5$, and the inside function is $g(x) = x^2 + 8$.

$$f(x) = x^5 \qquad f'(x) = 5x^4$$
$$g(x) = x^2 + 8 \qquad g'(x) = 2x$$

The derivative of the outside function, evaluated at the inside function, and then multiplied by the derivative of the inside function is:

$$\frac{dy}{dx} = 5(x^2+8)^4(2x).$$

The Wolfram Alpha syntax is: `derivative of (x^2+8)^5`.

Example 3 Use the chain rule to differentiate $y = \sin(\sqrt{x})$.

The outside function is $f(x) = \sin x$, and the inside function is $g(x) = \sqrt{x}$.

$$\begin{array}{ll} f(x) = \sin x & f'(x) = \cos x \\ g(x) = \sqrt{x} & g'(x) = \frac{1}{2}x^{-1/2} \end{array}$$

The derivative of the outside function, evaluated at the inside function, and then multiplied by the derivative of the inside function is:

$$\frac{dy}{dx} = \cos(\sqrt{x}) \cdot (\frac{1}{2}x^{-1/2})$$

The Wolfram Alpha syntax is: `derivative of sin(sqrt(x))`.

Example 4 Find the slope of the line tangent to the graph of $F(x) = \sqrt[3]{(x^2-1)^2}$ at the point where $x = 3$.

We must compute $F'(3)$ and in order to do so, we need the chain rule. In the function F, we see that $x^2 - 1$ is "plugged into" the radical function $x^{2/3}$ The outside function is $f(x) = x^{2/3}$, while the inside function is $g(x) = x^2 - 1$.

$$\begin{array}{ll} f(x) = x^{2/3} & f'(x) = \frac{2}{3}x^{-1/3} \\ g(x) = x^2 - 1 & g'(x) = 2x \end{array}$$

The derivative of the outside function, evaluated at the inside function, and then multiplied by the derivative of the inside function is:

$$F'(x) = \frac{2}{3}(x^2-1)^{-1/3} \cdot (2x) \text{ and } F'(3) = 2.$$

The Wolfram Alpha syntax is: `derivative of (x^2-1)^(2/3) at 3`.

Example 5 Let $f(x) = \sin(\cos 5x)$. Find $f'(x)$.

The outside function is $\sin x$ and the inside function is $\cos 5x$. The derivative of the outside function, evaluated at the inside function, and then multiplied by the derivative of the inside function is:

$$f'(x) = \cos(\cos 5x)\frac{d}{dx}\cos 5x.$$

There is an extra complication here: in order to find the derivative of the inside function, we need the chain rule.

$$\frac{d}{dx}\cos 5x = (-\sin 5x)(5)$$

After all is said and done, we have

$$f'(x) = \cos(\cos 5x)(-\sin 5x)(5).$$

The Wolfram Alpha syntax is: `derivative of sin(cos(5x))`.

It is not uncommon for the outside function of a composition to be a function of the form $y = x^n$. This class of compositions can be differentiated by using a special case of the chain rule called the general power rule.

General Power Rule

$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \cdot f'(x)$

Example 6 Use the general power rule to find dy/dx if $y = \tan^5 x$.

$$\frac{dy}{dx} = 5(\tan x)^4(\sec^2 x)$$

Example 7 Let $f(x) = \cos x^2$. Find $f''(0)$.

The general power rule doesn't apply here. We use the standard chain rule.

$$f'(x) = (-\sin x^2)(2x)$$

$$f''(x) = (-\sin x^2)(2) + (2x)(-\cos x^2)(2x) = -4x^2 \cos x^2 - 2\sin x^2$$

$$f''(0) = 0$$

The Wolfram Alpha syntax is: `2nd derivative of cos(x^2) at 0`.

Example 8 Find the x-coordinate of each point at which the graph of $f(x) = x\sqrt{1-x^2}$ has a horizontal tangent line.

Notice that the domain of f is $[-1, 1]$, so we are only interested in x-values satisfying $-1 < x < 1$.

Horizontal tangent lines will occur at points where the derivative is zero.

$$f'(x) = x\left(\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right) + \sqrt{1-x^2} = (1-x^2)^{-1/2}\left(-x^2 + 1 - x^2\right)$$

$$(1-x^2)^{-1/2}(1-2x^2) = 0 \Longrightarrow 1 - 2x^2 = 0 \Longrightarrow x = \pm\frac{1}{\sqrt{2}}$$

If we use Wolfram Alpha to find $f'(x)$, the output will automatically include the zeros of f'.

Example 9 Suppose $y = 4u^7$ and $u = 3x^2 - 4$. Notice that y is a function of x through the action of the intermediate variable u. Use the first form of the chain rule to find dy/dx when $x = 1$.

Solution omitted.