# Lecture 18: Extreme values on closed and bounded intervals

Objectives:

**(18.1)** Find the critical numbers of a function.

**(18.2)** Determine whether an extreme value is absolute or relative.

**(18.3)** Find the absolute extreme values of a continuous function on $[a, b]$.

## Absolute extrema

One of the most common applications of the derivative is in the optimization of functions. To optimize a function means to find its minimum and maximum values. The minimum and maximum values are collectively called extreme values or extrema. There are two different kinds of extrema: absolute (or global) and relative (or local).

**Definition of absolute extrema**

Suppose $f$ is defined on a set $D$ containing the number $c$.

- $f(c)$ is the absolute maximum value of $f$ on $D$ if $f(c) \geq f(x)$ for all $x$ in $D$.
- $f(c)$ is the absolute minimum value of $f$ on $D$ if $f(c) \leq f(x)$ for all $x$ in $D$.

According to this definition, a function's absolute extrema are the overall greatest and least function values. Not every function has absolute extreme values. For example, the function $f(x) = x$ takes on neither a minimum value nor a maximum value. On the other hand, $g(x) = x^2$ attains a minimum value (at $x = 0$) but no maximum value. The following theorem describes an important class of functions—functions that are guaranteed to attain both a minimum value and a maximum value.

**Extreme Value Theorem**

If $f$ is continuous on the closed and bounded interval $[a, b]$, then $f$ attains both an absolute minimum value and an absolute maximum value on $[a, b]$.

## Relative extrema

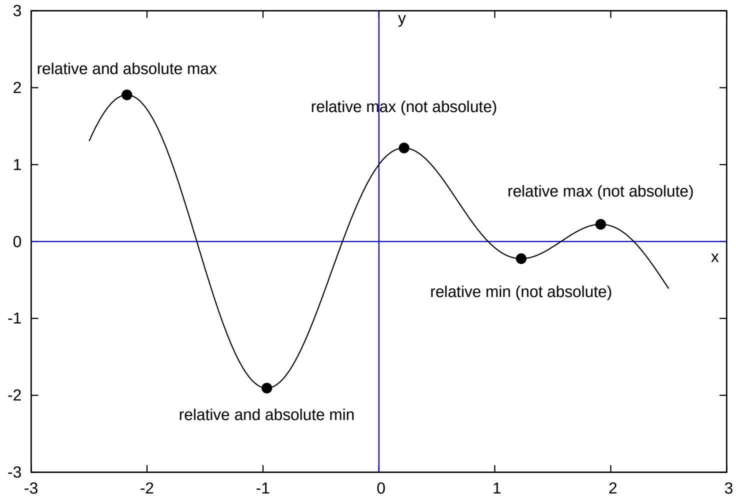
Knowing that extreme values exist is only half the battle, the other half is to find them. Before we get around to locating extreme values, we need to discuss the second kind of extrema.

**Definition of relative extrema**

Suppose $f$ is a function with domain $D$.

- If, inside $D$, there is an open interval containing $c$ on which $f(c)$ is an absolute maximum, then $f(c)$ is a relative maximum.
- If, inside $D$, there is an open interval containing $c$ on which $f(c)$ is an absolute minimum, then $f(c)$ is a relative minimum.

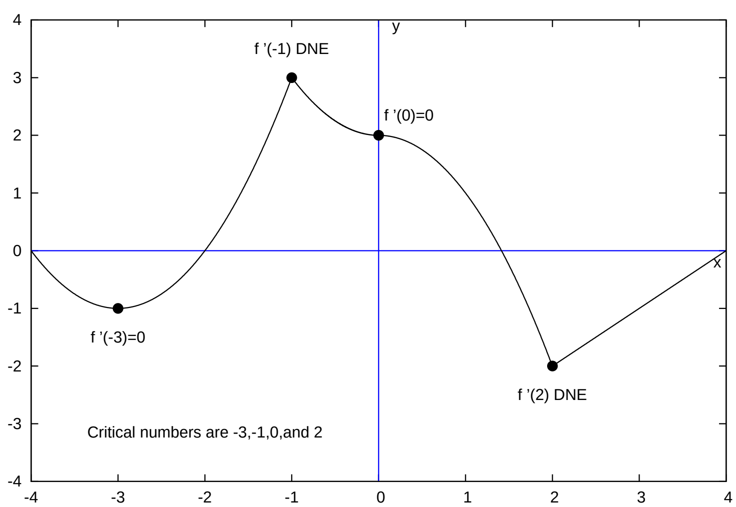
The figure below illustrates both absolute and relative extrema.




According to the definition, relative extrema can only occur at domain interior points[1]. A somewhat unsettling consequence is that not all absolute extrema are relative extrema. For example, an absolute maximum that occurs at a domain endpoint is not considered a relative maximum. We won't let this bother us as we turn to the task of locating extrema.

> **Definition of critical number**
>
> If $c$ is an interior point in the domain of $f$ at which $f'(c) = 0$ or $f'(c)$ does not exist, then $c$ is called a critical number (or critical point) of $f$.

The figure below illustrates the two types of critical numbers.




**Example 1** Find the critical numbers of $f(x) = 2x^3 + 3x^2 - 12x + 6$.

$$f'(x) = 6x^2 + 6x - 12 = 6(x-1)(x+2)$$
$$f'(x) = 0 \Longrightarrow x = 1 \text{ or } x = -2$$
$$f'(x) \text{ DNE never}$$

Since both 1 and $-2$ are domain interior points, the critical numbers are $x = 1$ and $x = -2$.

[1] The number $t$ is an interior point of the set $S$ if there exists an open interval $(a, b)$ containing $t$ that is entirely inside $S$.

**Example 2** Find the critical numbers of $h(x) = x\sqrt{x+1}$.

*Details omitted. The only critical number is $x = -2/3$ at which $h'(x) = 0$.*

The following theorem tells us why critical numbers are important, and it also leads to a procedure for locating extreme values.

> **Theorem 1 — Relative extrema and critical numbers**
>
> If $f$ takes on an extreme value at a domain interior point $c$, then $c$ is a critical number.

## Finding extrema on closed & bounded intervals

According to Theorem 1, extreme values at domain interior points must occur at critical numbers. The only other places where extreme values could possibly occur are at domain boundary points. Therefore, a list of all critical numbers and domain boundary points must include the locations of all extrema. This idea gives rise to the following procedure, which only applies to absolute extrema.

> **Finding absolute extrema**
>
> To find the absolute extreme values of $f$ on $[a, b]$:
>
> 1. Find all critical numbers in $(a, b)$.
> 2. Evaluate $f$ at all critical numbers and at the domain endpoints, $a$ and $b$.
> 3. Of the function values computed in step 2, the greatest is the absolute maximum, and the least is the absolute minimum.

**Example 3** Find the absolute extreme values of $f(x) = 3x^4 - 8x^3 - 48x^2 + 5$ on $[-3, 1]$.

We begin by finding all critical numbers.

$$f'(x) = 12x^3 - 24x^2 - 96x = 12x(x-4)(x+2)$$

$f'$ exists everywhere, and its zeros are $x = 0$, $x = 4$ and $x = -2$. Since 4 is outside the domain interval, it cannot be a critical number. Therefore, there are two critical numbers: $x = 0$ and $x = -2$. Next we evaluate $f$ at the critical numbers and the domain endpoints.

| $x$ | $0$ | $-2$ | $-3$ | $1$ |
|---|---|---|---|---|
| $f(x)$ | $5$ | $-75$ | $32$ | $-48$ |

The absolute maximum value is $f(-3) = 32$. The absolute minimum value is $f(-2) = -75$. The Wolfram Alpha syntax is: `maximize 3x^4-8x^3-48x^2+5 over [-3,1]` (or we can use `minimize`).

**Example 4** Find the absolute extreme values of $g(x) = (2x-4)^{2/3}$ on $[0, 5]$.

$g'(x) = \dfrac{4}{3\sqrt[3]{2x-4}}$. The only critical number is $x = 2$ where $g'$ does not exist.

| $x$ | $2$ | $0$ | $5$ |
|---|---|---|---|
| $f(x)$ | $0$ | $\sqrt[3]{16}$ | $\sqrt[3]{36}$ |

The absolute maximum value is $f(5) = \sqrt[3]{36}$. The absolute minimum value is $f(2) = 0$.