Lecture 23: Curve sketching

Objectives:

(23.1) Use algebra and calculus techniques to identify important features of a function and its graph.

(23.2) Sketch the graph of a function.

Curve sketching

We are already familiar with using a number of algebraic techniques to sketch the graph of a function. Among others, these techniques include plotting points; finding intercepts; testing for and using symmetry; using translations, reflections, and dilations; etc. Now with the calculus techniques we have recently studied, such as asymptotes and the 1st and 2nd derivative tests, we are prepared to sketch very accurate, detailed graphs of functions. Of course, it goes without saying that graphing calculators and computer algebra systems can be pretty helpful as well.

Example 1 Analyze and sketch the graph of $f(x) = x^4 - 4x^2$.

f is a polynomial function. It is continuously differentiable for all values of x. Since it has even degree and a positive leading coefficient, we know the graph points upward on both ends. The graph has no asymptotes. The y-intercept is $(0, f(0)) = (0,0)$. The x-intercepts are found by solving $f(x) = 0$:

$$x^4 - 4x^2 = 0 \Longrightarrow x^2(x-2)(x+2) = 0 \Longrightarrow x = 0,\ x = 2,\ x = -2$$

Therefore the x-intercepts are $(0,0)$, $(2,0)$, and $(-2,0)$. Since

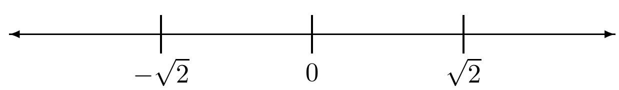
$$f(-x) = (-x)^4 - 4(-x)^2 = x^4 - 4x^2 = f(x),$$

the function is even, and it graph is symmetric about the y-axis.

Applying the 1st derivative test, we find that

$$f'(x) = 4x^3 - 8x = 0 \Longrightarrow 4x(x^2-2) = 0 \Longrightarrow x = 0,\ x = \sqrt{2},\ x = -\sqrt{2}.$$

The critical numbers are $x = 0$, $x = \sqrt{2}$, and $x = -\sqrt{2}$.



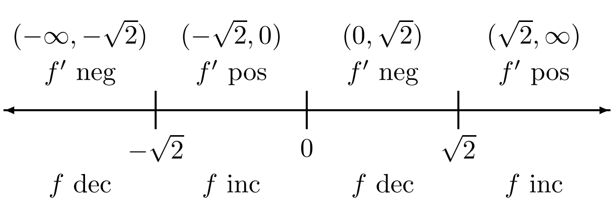
We now determine the sign of f' on each interval. Recall that $f'(x) = 4x(x^2-2)$.

$$x \text{ in } (-\infty, -\sqrt{2}) \Longrightarrow f'(x) < 0$$

$$x \text{ in } (-\sqrt{2}, 0) \Longrightarrow f'(x) > 0$$

$$x \text{ in } (0, \sqrt{2}) \Longrightarrow f'(x) < 0$$

$$x \text{ in } (\sqrt{2}, \infty) \Longrightarrow f'(x) > 0$$

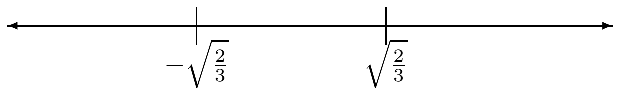


Based on the sign chart for f', we see that f has relative minimum values $f(-\sqrt{2}) = -4$ and $f(\sqrt{2}) = -4$ and relative maximum value $f(0) = 0$.

Applying the 2nd derivative test, we find that

$$f''(x) = 12x^2 - 8 = 0 \Longrightarrow x = -\sqrt{2/3},\ x = \sqrt{2/3}.$$

The Hergert numbers are $x = -\sqrt{2/3}$ and $x = \sqrt{2/3}$.

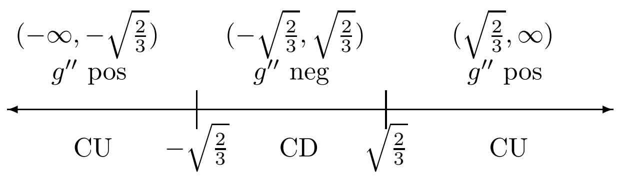


We now determine the sign of f'' on each interval. Recall that $f''(x) = 12x^2 - 8$.

$$x \text{ in } (-\infty, -\sqrt{2/3}) \Longrightarrow f''(x) > 0$$

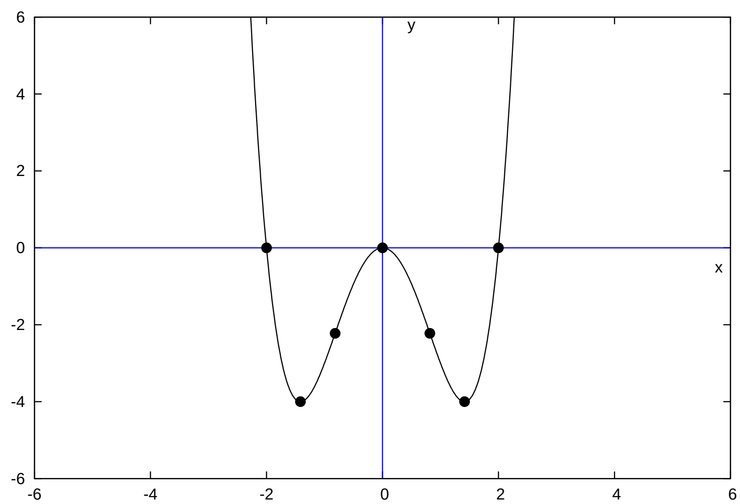
$$x \text{ in } (-\sqrt{2/3}, \sqrt{2/3}) \Longrightarrow f''(x) < 0$$

$$x \text{ in } (\sqrt{2/3}, \infty) \Longrightarrow f''(x) > 0$$



Based on the sign chart for f'', we see that the graph of f has inflection points at $x = -\sqrt{2/3}$ and $x = \sqrt{2/3}$. These are the points $(-\sqrt{2/3}, -20/9)$ and $(\sqrt{2/3}, -20/9)$.

Using all of the information found above, we now sketch the graph.



Example 2 Analyze and sketch the graph of $g(x) = \dfrac{x^2 + x + 25}{x + 1}$.

The details are omitted, but the analysis is summarized as follows.

Domain	$(-\infty, -1) \cup (-1, \infty)$
Symmetry	None
Intercepts	y-intercept is $(0, 25)$ No x-intercepts
Asymptotes	Vertical: $x = -1$ Slant: $y = x$
1st Derivative	$g'(x) = \dfrac{(x-4)(x+6)}{(x+1)^2}$ $g'(x) = 0 \Longrightarrow x = 4,\ x = -6$ $g'(x)$ DNE $\Longrightarrow x = -1$ g is increasing on $(-\infty, -6) \cup (4, \infty)$ g is decreasing on $(-6, -1) \cup (-1, 4)$ Relative minimum value is $g(4) = 9$ Relative maximum value is $g(-6) = -11$
2nd Derivative	$g''(x) = \dfrac{50}{(x+1)^3}$ $g''(x) = 0$ never $g''(x)$ DNE $\Longrightarrow x = -1$ Graph of g is CD on $(-\infty, -1)$ Graph of g is CU on $(-1, \infty)$ The graph has no inflection points.

Using the information above, we now sketch the graph.