Lecture 25: Linearizations

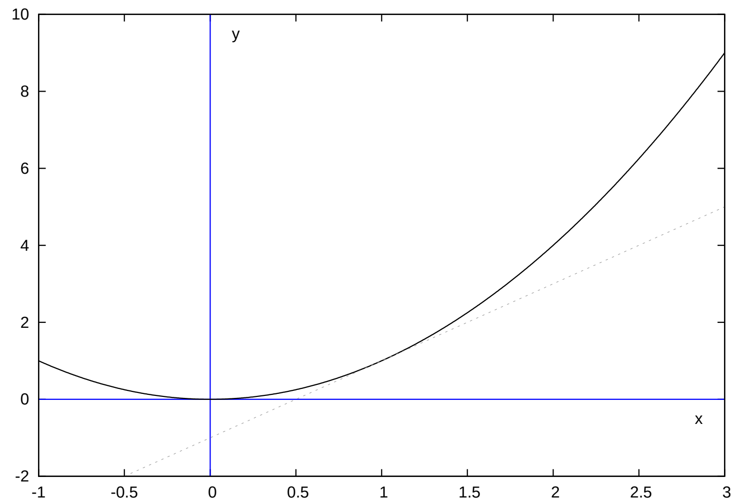
Objectives:

(25.1) Determine the linearization of a function at a point.

(25.2) Use the linearization at a point to approximate function values near the point.

Tangent lines and linearizations

A tangent line at a point on a smooth graph is typically very close to the graph near that point. For example, the graph of $y = x^2$ and its tangent line at $x = 1$ are shown below. Notice how close they are near $x = 1$.



Because of this typical behavior, it is quite common to use tangent lines to approximate functions. In fact, we will see some very interesting and useful applications of this idea.

As we have computed in earlier lectures, the line tangent to the graph of f at $x = c$ is described in point-slope form by the equation

$$y - f(c) = f'(c)(x - c).$$

This leads directly to our definition of linearization.

Definition of linearization
Suppose f is differentiable at $x = c$. The linearization of f at $x = c$ is the function

$$L(x) = f(c) + f'(c)(x - c).$$

The approximation $f(x) \approx L(x)$ is called the standard linear approximation of f at c.
It is often a good approximation near $x = c$.

Example 1 Find the linearization of $f(x) = \sin x$ at $x = 0$. Then use it to approximate $\sin(0.12)$.

Since $f'(x) = \cos x$, we have

$$L(x) = f(0) + f'(0)(x - 0) = \sin(0) + \cos(0)(x - 0) = x.$$

The linearization is $L(x) = x$. It follows that $\sin(0.12) \approx 0.12$. This is actually a very good approximation—the percent error is

$$\left| \frac{\sin(0.12) - 0.12}{\sin(0.12)} \right| \times 100\% \approx 0.24\%.$$

The standard linear approximation, $\sin x \approx x$ for small x, is very useful in many science and engineering applications.

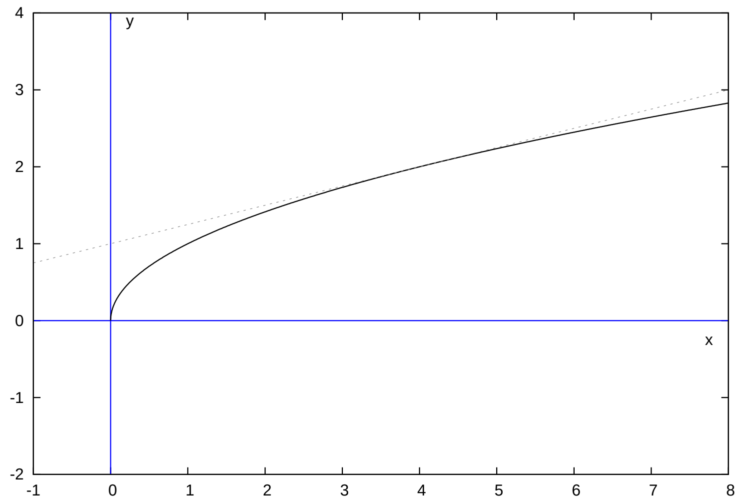
Example 2 Let $f(x) = \sqrt{x}$.

1. Find the linearization of f at $x = 4$.

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$L(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$$

2. Sketch the graph of $y = f(x)$ and its linearization at $x = 4$.



3. Use the linearization to approximate $\sqrt{3.98}$.

$$\sqrt{3.98} \approx L(3.98) = 2 + \frac{1}{4}(3.98 - 4) = 2 + (-0.005) = 1.995$$

Example 3 Suppose there is a function $y(x)$ that satisfies the following initial value problem:

$$\frac{dy}{dx} = \sqrt{y} + xy^2; \quad y(2) = 1.$$

Use the linearization of y at $x = 2$ to approximate $y(2.1)$.

The linearization of y at $x = 2$ is given by

$$L(x) = y(2) + y'(2)(x-2)$$

The value of $y(2)$ is given by the initial condition, $y(2) = 1$. $y'(2)$ follows from the derivative:

$$y'(2) = \frac{dy}{dx} \text{ at } (2,1) = \sqrt{1} + 2(1)^2 = 3.$$

Therefore, the linearization is given by

$$L(x) = 1 + 3(x-2),$$

and $y(2.1) \approx L(2.1) = 1.3$.

Example 4 Let $g(x) = x - \cos x$. Let $L(x)$ be the linearization of g at $x = 1$. Approximate the solution of $g(x) = 0$ by solving $L(x) = 0$.

Since $g'(x) = 1 + \sin x$, the linearization of g at $x = 1$ is given by

$$L(x) = g(1) + g'(1)(x-1) = (1 - \cos 1) + (1 + \sin 1)(x-1) = 0.459698 + 1.841471(x-1).$$

Setting $L(x) = 0$, we find that $x = 0.750364$. Our approximate solution of $g(x) = 0$ is $x \approx 0.750364$.