Lecture 28: Antiderivatives and indefinite integrals

Objectives:

(28.1) Use basic integration rules to evaluate indefinite integrals.

(28.2) Solve initial value problems.

Antiderivatives and indefinite integrals

Now that we have become experts at differentiation, it is time for us to think about antidifferentiation—the inverse of the differentiation operation. As a simple example, consider the function $f(x) = 2x$. If f is the end result of differentiating a function, what was the original function? Because we are good at differentiating, we immediately recognize that

$$\text{derivative} = 2x \quad \Longrightarrow \quad \text{original function} = x^2.$$

We will call $F(x) = x^2$ an antiderivative of $f(x) = 2x$. As is the case with all of the inverse operations we have studied in mathematics, the inverse of differentiation is just as important as differentiation itself.

Definition of antiderivative

The function F is an antiderivative of the function f on the interval I if $F'(x) = f(x)$ for all x in I.

Example 1 Find an antiderivative of the function $f(x) = \cos x$.

We are looking for a function whose derivative is $\cos x$. Because of our experience with differentiation, we immediately recognize that such a function is $F(x) = \sin x$. However, this not the only such function. Since the derivative of a constant is zero, any constant could be added to $\sin x$, and the derivative would still be $\cos x$. For example, $F(x) = 5 + \sin x$ or $F(x) = -13.72 + \sin x$.

In the example above, we noticed that the antiderivative of a function is not unique: if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C. Are there antiderivatives that do not differ by a constant? The answer is no, as stated in the following theorem.

Theorem 1 — Functions with identical derivatives

Two functions have the same derivative on the interval $[a, b]$ if and only if the functions differ by a constant. That is, on the interval $[a, b]$,

$$F'(x) = G'(x) \quad \text{iff} \quad F(x) = G(x) + C.$$

According to Theorem 1, antiderivatives on an interval are unique up to a constant. Therefore, to find all antiderivatives of a function, we simply need to find one antiderivative. Any other antiderivative can be obtained from the first by the addition of a constant.

Definition of indefinite integral

Suppose F is a function such that $F'(x) = f(x)$. The indefinite integral of f, denoted by $\int f(x)\,dx$, is the most general antiderivative of f, which includes an arbitrary *constant of integration*:

$$\int f(x)\,dx = F(x) + C.$$

The indefinite integral gives a generic description of every possible antiderivative.

For every differentiation fact, there is a corresponding antidifferentiation fact. An antidifferentiation (or integration) formula can be obtained by simply reading a differentiation fact backward. For example,

$$\frac{d}{dx}x = 1 \qquad \Longrightarrow \qquad \int 1\,dx = x + C$$

$$\frac{d}{dx}\sin x = \cos x \qquad \Longrightarrow \qquad \int \cos x\,dx = \sin x + C$$

$$\frac{d}{dx}(-\cos x) = \sin x \qquad \Longrightarrow \qquad \int \sin x\,dx = -\cos x + C$$

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = x^n \qquad \Longrightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

Example 2 Evaluate each indefinite integral by reversing a well-known differentiation fact.

1. $\int x^4\,dx$

   The power rule for differentiation tells us to multiply by the exponent and then decrease the exponent by 1. Therefore, the power rule for integration must tell us (as above) to increase the exponent by 1 and then divide by the exponent.

$$\int x^4\,dx = \frac{x^5}{5} + C$$

2. $\int \sec^2 x\,dx$

$$\int \sec^2 x\,dx = \tan x + C$$

3. $\int \frac{1}{x^3}\,dx$

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = -\frac{1}{2x^2} + C$$

Reversing differentiation facts is not always as easy as the previous examples make it seem. The next theorem helps us to integrate functions that are slightly more complicated.

Theorem 2 — Properties of indefinite integrals

Suppose that antiderivatives of f and g exist.

- $\int k\,f(x)\,dx = k\int f(x)\,dx$ for any constant k
- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

When integrating a sum or difference of functions by using the properties above, it is not necessary to add a constant of integration for each function. The sum or difference of arbitrary constants is still just an arbitrary constant. Therefore, one constant of integration is sufficient.

Example 3 Evaluate each indefinite integral.

1. $\int (3x^5 + 2x + 7)\,dx$

   $$\int (3x^5 + 4x + 7)\,dx = 3\int x^5\,dx + 4\int x\,dx + 7\int 1\,dx = 3\left(\frac{x^6}{6}\right) + 4\left(\frac{x^2}{2}\right) + 7x + C = \frac{1}{2}x^6 + 2x^2 + 7x + C$$

2. $\int (5\sin t + 8\cos t)\,dt$

   $$\int (5\sin t + 8\cos t)\,dt = 5(-\cos t) + 8\sin t + C = -5\cos t + 8\sin t + C$$

3. $\int \frac{\sqrt{x} + (2x+3)^2}{x^5}\,dx$

   $$\int \frac{\sqrt{x} + (2x+3)^2}{x^5}\,dx = \int \frac{x^{1/2} + 4x^2 + 12x + 9}{x^5}\,dx = \int (x^{-9/2} + 4x^{-3} + 12x^{-4} + 9x^{-5})\,dx = -\frac{2}{7}x^{-7/2} - 2x^{-2} - 4x^{-3} - \frac{9}{4}x^{-4} + C$$

Initial value problems

An equation involving the derivative of an unknown function is called a *differential equation*. To solve a differential equation is to find the unknown function. A differential equation of the form

$$\frac{dy}{dx} = f(x)$$

can be solved by antidifferentiating the function f. Since the antidifferentiation will involve the addition of a constant of integration, the solution of the differential equation is actually an entire family of solutions. A particular solution can be identified by giving an *initial condition*, $y(x_0) = y_0$. The problem of solving a differential equation subject to an initial condition,

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0,$$

is called an *initial value problem*.

Example 4 Solve the initial value problem: $\frac{dy}{dx} = 6x^2 + 4x, \quad y(0) = 5.$

$$y = \int \frac{dy}{dx}\,dx = \int (6x^2 + 4x)\,dx = 2x^3 + 2x^2 + C$$

So $y(x) = 2x^3 + 2x^2 + C$ for some constant C. We need only find the specific C that satisfies $y(0) = 5$.

$$y(0) = 5 \Longrightarrow 2(0)^3 + 2(0)^2 + C = 5 \Longrightarrow C = 5$$

The solution is $y(x) = 2x^3 + 2x^2 + 5$.

Example 5 Determine the function f such that $f'(x) = 8x + \cos x$ and $f(\pi) = \pi^2$.

The details are omitted, but we should find that $f(x) = 4x^2 + \sin x - 3\pi^2$.

Indefinite integrals in a CAS

Computer algebra systems such as Wolfram Alpha and Maxima have very sophisticated built-in functions for indefinite integration.

Example 6 Use Wolfram Alpha to evaluate $\int (\sqrt{x} + 5\cos x)\,dx$.

The Wolfram Alpha syntax is `integrate (sqrt(x)+5*cos(x))`. The result should look something like

$$\frac{2x^{3/2}}{3} + 5\sin(x) + \text{constant}.$$

Example 7 Use Maxima to evaluate $\int \left(\frac{5}{t^6} + \sqrt[5]{t^3}\right)\,dt$.

The Maxima syntax is `integrate( 5/t^6 + t^(3/5), t );`. The result should look something like

$$\frac{5t^{8/5}}{8} - \frac{1}{t^5}.$$

Notice that Maxima does not include the constant of integration.

Example 8 Computer algebra systems can also solve initial value problems. Use Maxima to solve the initial value problem from Example 5.

Maxima requires two commands, but they can be placed on the same line. The syntax is `ode2( 'diff(y,x) = 8*x+cos(x), y, x ); ic1( %, x=%pi, y=%pi^2 );`. The solution should look something like

$$y = \sin(x) + 4x^2 - 3\pi^2.$$