Lecture 29: Area, lower sums, and upper sums

Objectives:

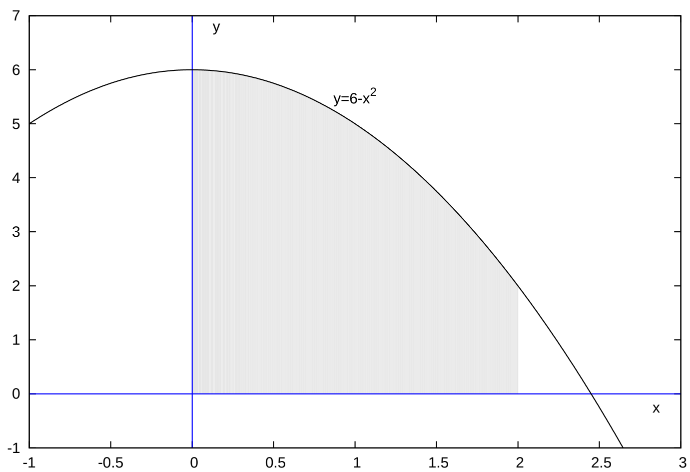
(29.1) Use rectangles to approximate the area of a bounded region under the graph of a function.

(29.2) Use upper and lower sums to approximate area.

Areas by rectangles

In this lecture we will begin our study of area under a curve. This will lead us down the path toward the definite integral and the Fundamental Theorem of Calculus, but those are several lectures away. For now, let's consider the following problem.

Problem: Use rectangles of equal base length to approximate the area of the region bounded by the graphs of $f(x) = 6 - x^2$, $y = 0$, $x = 0$, and $x = 2$. The bounded region is shown below.

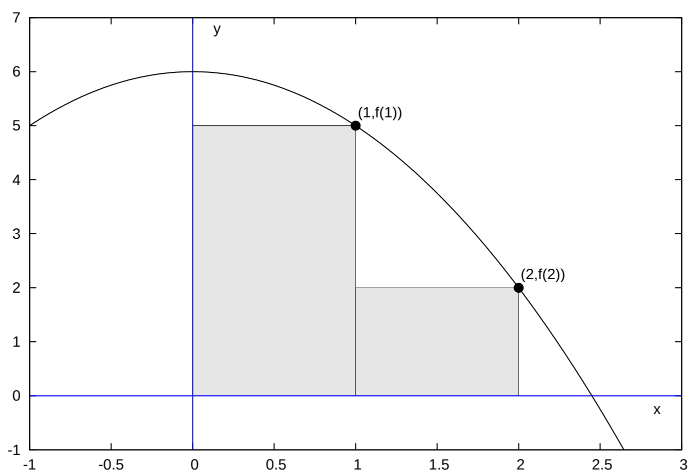


Solution 1 Solve the problem above by using 2 rectangles that lie entirely under the curve.

Since the two rectangles must have equal base lengths, the length will be

$$\Delta x = \frac{2-0}{2} = 1.$$

The rectangles are supposed to lie entirely under the curve. Therefore, the best such approximation will come from rectangles that just reach high enough to touch the graph of $f(x) = 6 - x^2$. The rectangles are shown below. Because of the shape of the curve, the heights of the rectangles are determined by the values of $f(x)$ at the rectangles' right sides.



The area of the region can now be approximated by computing the sum of the areas of the rectangles:

$$\text{Area} \approx \text{Area of 1st rectangle} + \text{Area of 2nd rectangle}$$

or

$$\text{Area} \approx 1 \cdot f(1) + 1 \cdot f(2) = 5 + 2 = 7.$$

Because the rectangles lie entirely below the curve, we will call this a *lower sum.* Our lower sum under-estimates the area of the region. After letting $c_1 = 1$ and $c_2 = 2$, notice that our area approximation has the form

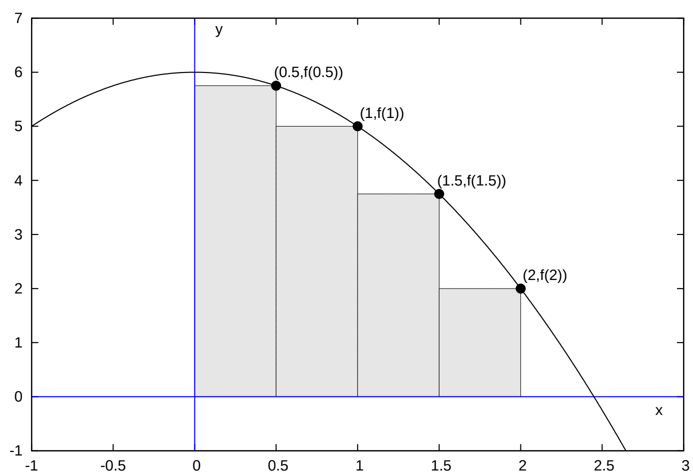
$$\text{Area} \approx \sum_{k=1}^{2} f(c_k)\Delta x.$$

Solution 2 Solve the problem above by using 4 rectangles that lie entirely under the curve.

If we use 4 rectangles of equal base length, we have

$$\Delta x = \frac{2-0}{4} = \frac{1}{2} = 0.5.$$

As above, the heights of the rectangles are determined by the function values at the rectangles' right sides.



If we let $c_1 = 0.5$, $c_2 = 1$, $c_3 = 1.5$ and $c_4 = 2$, then our area approximation has the form

$$\text{Area} \approx \sum_{k=1}^{4} f(c_k)\Delta x$$

or

$$\text{Area} \approx 0.5 \cdot (f(0.5) + f(1) + f(1.5) + f(2)) = 8.25.$$

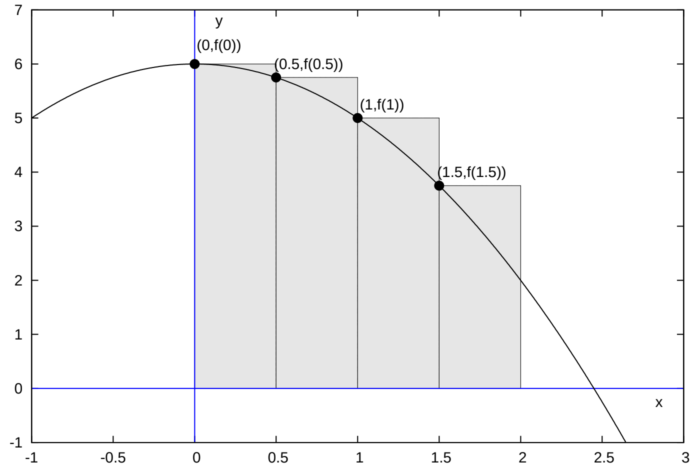
Once again this is a lower sum, and it under-estimates the actual area of the region.

Solution 3 Solve the problem above by using 4 rectangles that lie entirely above the curve.

As above, we have

$$\Delta x = \frac{2-0}{4} = \frac{1}{2} = 0.5.$$

In this case, however, the heights of the rectangles are determined by the function values at the rectangles' **left** sides.



If we let $c_1 = 0$, $c_2 = 0.5$, $c_3 = 1$ and $c_4 = 1.5$, then our area approximation has the form

$$\text{Area} \approx \sum_{k=1}^{4} f(c_k)\Delta x$$

or

$$\text{Area} \approx 0.5 \cdot (f(0) + f(0.5) + f(1) + f(1.5)) = 10.25.$$

This time we have computed an *upper sum*. It over-estimates the actual area of the region.

By using the GeoGebra applet available at `http://stevekifowit.com/geo_apps`, we can experiment with the lower and upper sums associated with $f(x) = 6 - x^2$ on $[0, 2]$.

Solution 4 Instead of using a fixed number of rectangles, derive a formula that gives the lower sum associated with n rectangles of equal base length.

This is a tough problem! First, we determine the rectangle base length:

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

The heights of the rectangles will be determined by the values of $f(x)$ at the right sides of the rectangles. The right sides will occur at the following x-values:

$$\frac{2}{n}, \frac{4}{n}, \frac{6}{n}, \ldots, \frac{2n}{n}.$$

Because we will be evaluating f at these values, we need a general formula for right-hand side of the kth rectangle. In this case, it is easy to see that the right side of the kth rectangle occurs at

$$x = c_k = \frac{2k}{n}, \text{ for } k = 1, 2, 3, \ldots n.$$

The area approximation is now given by

$$Area \approx LS(n) = \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} \left[6 - \left(\frac{2k}{n}\right)^2\right] \frac{2}{n}.$$

Instead of taking the time to algebraically simplify this expression, we will let a CAS do the work for us. According to Wolfram Alpha, using `sum (6-(2*k/n)^2)*2/n, k=1 to n`,

$$LS(n) = \sum_{k=1}^{n} \left[6 - \left(\frac{2k}{n}\right)^2\right] \frac{2}{n} = \frac{4(7n^2 - 3n - 1)}{3n^2}.$$

Notice the $LS(2) = 7$ and $LS(4) = 8.25$ as we computed above.

Solution 5 Use an approach similar to that of Solution 4 to derive a formula that gives the upper sum associated with n rectangles of equal base length.

As above the rectangle base length is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

In this case, the heights of the rectangles will be determined by the values of $f(x)$ at the left sides of the rectangles. The left sides will occur at the following x-values:

$$\frac{0}{n}, \frac{2}{n}, \frac{4}{n}, \ldots, \frac{2n-2}{n}.$$

The left side of the kth rectangle is described by the formula

$$x = c_k = \frac{2k-2}{n}, \text{ for } k = 1, 2, 3, \ldots, n.$$

The area approximation is now given by

$$Area \approx US(n) = \sum_{k=1}^{n} f(c_k)\Delta x = \sum_{k=1}^{n} \left[6 - \left(\frac{2k-2}{n}\right)^2\right] \frac{2}{n}.$$

According to Wolfram Alpha, using `sum (6-(2*k-2)^2/n^2)*2/n, k=1 to n`,

$$US(n) = \sum_{k=1}^{n} \left[6 - \left(\frac{2k-2}{n}\right)^2\right] \frac{2}{n} = \frac{4(7n^2+3n-1)}{3n^2}.$$

Notice the $US(4) = 10.25$ as we computed above.

Lower and upper sums

At this point we'll set aside the problem from above and focus on lower and upper sums in general.

Suppose f is a nonnegative, continuous function on the interval $[a, b]$. Partition $[a, b]$ into n subintervals of equal length:

$$\Delta x = \frac{b-a}{n}.$$

The endpoints of the subintervals are described as follows:

$$a = x_0 < x_1 = a + \Delta x < x_2 = a + 2\Delta x < \cdots < x_k = a + k\Delta x < \cdots < x_n = a + n\Delta x = b.$$

The kth subinterval is the subinterval that ends at point x_k, namely $[x_{k-1}, x_k]$. On each subinterval, define m_k and M_k as follows:

$$m_k = x\text{-value at which } f(x) \text{ attains a minimum on } [x_{k-1}, x_k]$$

$$M_k = x\text{-value at which } f(x) \text{ attains a maximum on } [x_{k-1}, x_k]$$

Notice that, since f is a continuous function, m_k and M_k must exist on each subinterval. It now follows that the lower sum for f on $[a, b]$ using n rectangles is given by

$$LS(n) = \sum_{k=1}^{n} f(m_k)\Delta x$$

and the corresponding upper sum is given by

$$US(n) = \sum_{k=1}^{n} f(M_k)\Delta x.$$

Theorem 1 — Limits of lower and upper sums

Suppose f is a nonnegative, continuous function on $[a, b]$. Let n be a positive integer, and define the lower sum, $LS(n)$, and the upper sum, $US(n)$, as above. Then

$$\lim_{n\to\infty} LS(n) = \lim_{n\to\infty} US(n).$$

Remember that the lower and upper sums give approximations for the area of the region bounded by the graphs of $y = f(x)$, $y = 0$, $x = a$, and $x = b$. It is clear from their definitions that

$$LS(n) \leq \text{Actual area} \leq US(n)$$

for each positive integer n. It follows from the squeeze theorem (Lecture 5) that the actual area must be equal to the limit of the lower or upper sums.

Example 1 Referring back to Solution 4 or Solution 5 (above), find the actual area of the region bounded by the graphs of $y = 6 - x^2$, $y = 0$, $x = a$, and $x = b$.

$$\text{Area} = \lim_{n\to\infty} LS(n) = \lim_{n\to\infty} \frac{4(7n^2 - 3n - 1)}{3n^2} = \frac{28}{3}$$

Just as a check, notice that we also have

$$\lim_{n\to\infty} US(n) = \lim_{n\to\infty} \frac{4(7n^2 + 3n - 1)}{3n^2} = \frac{28}{3}.$$

We now make one more observation. Referring back to the definitions of the lower and upper sums, for each k, let c_k be any x-value chosen from the kth subinterval. It follows that

$$f(m_k) \leq f(c_k) \leq f(M_k), \text{ for } k = 1, 2, 3, \ldots n.$$

Therefore, we must have

$$LS(n) = \sum_{k=1}^{n} f(m_k)\Delta x \leq \sum_{k=1}^{n} f(c_k)\Delta x \leq \sum_{k=1}^{n} f(M_k)\Delta x = US(n).$$

The squeeze theorem applies once again to give the following result.

Theorem 2 — Area as a limit

Suppose f is a nonnegative, continuous function on $[a, b]$. Partition $[a, b]$ into n subintervals of equal length, $\Delta x = (b - a)/n$. For each k from 1 to n, let c_k be any point in the kth subinterval. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x.$$