Lecture 30: Riemann sums and the definite integral

Objectives:

(30.1) Compute a Riemann sum for a function on an interval.

(30.2) Define and interpret the definite integral of a function on an interval.

(30.3) Use Riemann sums to approximate definite integrals.

Riemann sums

In the last lecture we fell just a bit short of posing a very important definition—that of a Riemann sum. We'll continue with that goal in mind.

Suppose f is defined at each point of the interval $[a, b]$. Partition $[a, b]$ into n subintervals. In contrast to Lecture 29, f can take on positive or negative values, and our subintervals need not have equal length. The endpoints of the subintervals are given the generic names $x_0, x_1, x_2, \ldots, x_n$:

$$a = x_0 < x_1 < x_2 < \cdots < x_k < \cdots < x_n = b.$$

For $k = 1, 2, 3, \ldots, n$, let Δx_k be the length of the kth subinterval. That is,

$$\Delta x_k = x_k - x_{k-1} \quad \text{and} \quad x_k = x_{k-1} + \Delta x_k.$$

Now choose a point from each one of the subintervals. Give the name c_k to the point chosen from the kth subinterval. We now form the sum

$$\sum_{k=1}^{n} f(c_k)\Delta x_k.$$

This sum is called a *Riemann sum*. The definition is summarized below.

Definition of Riemann sum

Suppose f is defined on $[a, b]$. Partition $[a, b]$ into n subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let Δx_k be the length of the kth subinterval. The sum

$$\sum_{k=1}^{n} f(c_k)\Delta x_k, \quad x_{k-1} \le c_k \le x_k$$

is called a Riemann sum for f with the given partition.

Notice that the Riemann sum depends on the function f, the partition, and the choice of c_k's.

Example 1 Use 4 subintervals, all of different lengths, to compute a Riemann sum for $f(x) = 6 - x^2$ on the interval $[0, 2]$. Sketch the graph of f and the rectangles corresponding to your Riemann sum.

Let's use the following partition:

$$x_0 = 0 < x_1 = 0.4 < x_2 = 1.0 < x_3 = 1.7 < x_4 = 2.$$

For this partition we have

$$\Delta x_1 = 0.4, \quad \Delta x_2 = 0.6, \quad \Delta x_3 = 0.7, \quad \Delta x_4 = 0.3.$$

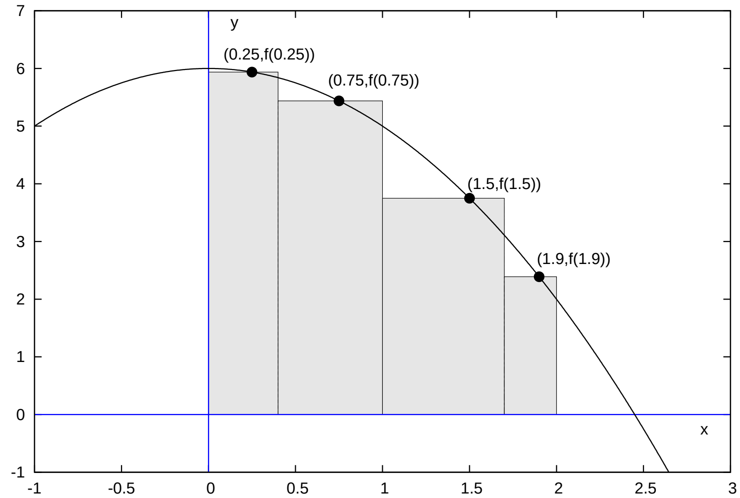
Now let's choose

$$c_1 = 0.25, \quad c_2 = 0.75, \quad c_3 = 1.5, \quad c_4 = 1.9.$$

Our Riemann sum is

$$\sum_{k=1}^{4} f(c_k)\Delta x_k = f(0.25) \cdot 0.4 + f(0.75) \cdot 0.6 + f(1.5) \cdot 0.7 + f(1.9) \cdot 0.3 = 8.9795.$$

The shaded rectangles corresponding to this Riemann sum are shown below.



As we can see from the example above, Riemann sums are far more general than lower and upper sums. Nonetheless they can be used in many of the same ways.

Example 2 Use a Riemann sum over 8 subintervals to approximate the area under the graph of $f(x) = x - x^3$ on $[0, 1]$.

We will use 8 subintervals of equal length $\Delta x = \frac{1}{8} = 0.125$:

$$x_0 = 0 < x_1 = 0.125 < x_2 = 0.25 < x_3 = 0.375 < x_4 = 0.5 < x_5 = 0.625 < x_6 = 0.75 < x_7 = 0.875 < x_8 = 1.$$

For each k, choose c_k to be the right endpoint of the kth subinterval. This gives the following Riemann sum

$$0.125 \cdot [f(0.125) + f(0.25) + f(0.375) + f(0.5) + f(0.625) + f(0.75) + f(0.875) + f(1)] = 0.24609375.$$

The actual area is 0.25.

Example 3 Use 4 subintervals of equal length and subinterval midpoints to compute a Riemann sum for $g(x) = 3 - 5x$ on $[0, 2]$.

In this example we have

$$\Delta x = \frac{2-0}{4} = \frac{1}{2} = 0.5,$$

and our partition is

$$x_0 = 0 < x_1 = 0.5 < x_2 = 1 < x_3 = 1.5 < x_4 = 2.$$

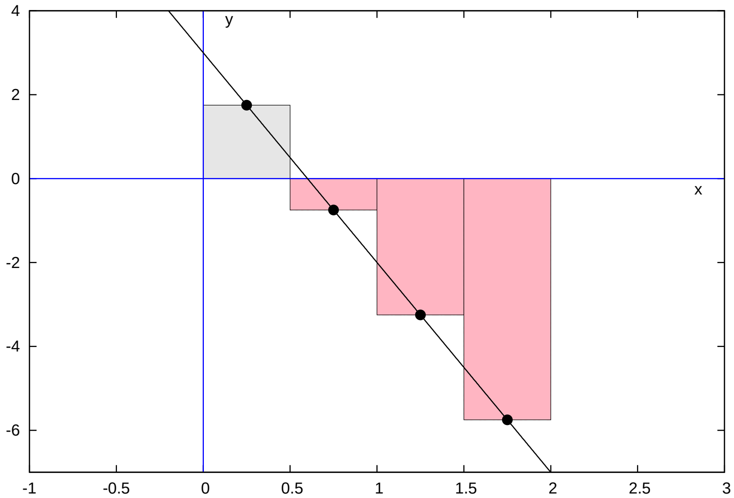
Using subinterval midpoints for the c_k's, we have

$$c_1 = 0.25, \quad c_2 = 0.75, \quad c_3 = 1.25, \quad c_4 = 1.75.$$

The corresponding Riemann sum is

$$\sum_{k=1}^{4} g(c_k)\Delta x = 0.5 \cdot [g(0.25) + g(0.75) + g(1.25) + g(1.75)] = -4.$$

The shaded rectangles associated with this sum are shown below. Notice that the Riemann sum is negative because some of the function values are negative. When computing the sum, rectangles below the x-axis (i.e. $f(c_k) < 0$) contribute negatively.



Riemann sums in a CAS

A GeoGebra applet for computing Riemann sums is available at `http://stevekifowit.com/geo_apps`. A Maxima program for computing Riemann sums is given below. The user has three options for the c_k's. Use `opt: 0` for left subinterval endpoints, `opt: 1` for subinterval midpoints, and `opt: 2` for right subinterval endpoints.

```
/* Riemann Sums */
f: sin(x)   $
a: 0   $
b: %pi   $
N: 20   $
opt: 2   $

h: float( b - a ) / N$
t: float( a + opt*h/2 )$
sum: 0.0$
for i: 1 thru N do
(  sum: sum + ev( f, numer, x=t ),
   t: t + h
)$
print( "Riemann sum is: ", h * sum )$
```

The definite integral

We are now ready for the definition of the definite integral. As we will see, the definite integral is defined to be a limit of Riemann sums.

Definition of the definite integral

Suppose f is defined on $[a, b]$. Partition $[a, b]$ into n subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let Δx_k be the length of the kth subinterval. Further let $||\Delta|| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$. The *definite integral* of f on $[a, b]$ is given by

$$\int_a^b f(x)\,dx = \lim_{||\Delta|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k, \quad x_{k-1} \le c_k \le x_k,$$

provided the limit exists. When the limit exists, f is said to be *Riemann integrable* on $[a, b]$. The function f is called the *integrand*.

Example 4 Use a Riemann sum over 12 subintervals to approximate $\int_1^4 \sqrt{x}\,dx$.

Using subintervals of equal length and subinterval midpoints as the c_k's, we find

$$\int_1^4 \sqrt{x}\,dx \approx 4.667.$$

The details are omitted.

Example 5 Refer back to Lecture 29 and find the value of $\int_0^2 (6-x^2)\,dx$.

In Lecture 29, we computed upper and lower sums for the function $f(x) = 6 - x^2$ on $[0, 2]$. In Example 1 of Lecture 29, we computed the area of the region under the graph of f by taking a limit of the upper and lower sums. In other words, we computed the limit of a Riemann sum. We found that

$$\int_0^2 (6-x^2)\,dx = \frac{28}{3}.$$

Now that we have defined the Riemann (definite) integral, it is worth considering what kinds of functions are Riemann integrable. A description of the entire class of Riemann integrable functions was first given by Henri Lebesgue, but his work is best left for an advanced math course. We will look at some simpler results.

Theorem — Continuous functions are integrable

If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

Important point

Not all functions on $[a, b]$ are integrable. For example, the characteristic function of the rational numbers,

$$\chi(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

is not Riemann integrable on $[0, 1]$.