Lecture 31: Properties of the definite integral

Objectives:

(31.1) Use properties of the definite integral to simplify and evaluate integrals.

(31.2) Use area to evaluate definite integrals.

(31.3) Use the definite integral to define the average value of a function.

Definite integrals and area

In this lecture we will study a number of important properties of the definite integral. First we focus on area.

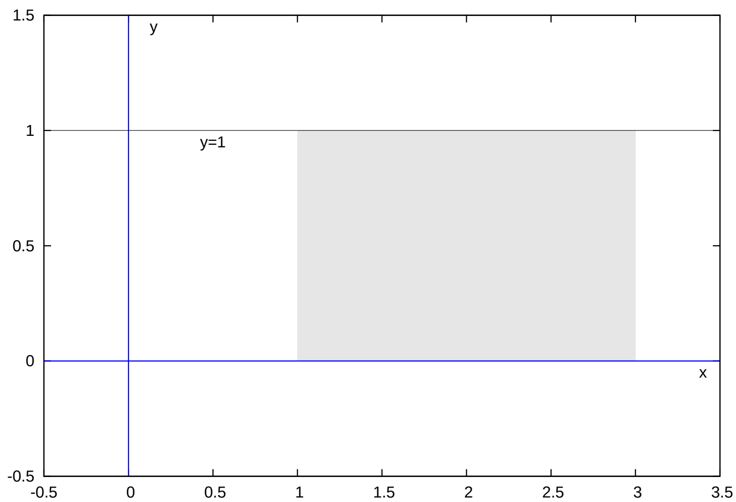
Theorem 1 — Area as a definite integral

Suppose f is a nonnegative, continuous function on $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

Example 1 Use geometry to evaluate $\int_1^3 1\,dx$.

To find the value of the integral, we can find the area of the region bounded by the graphs of $y = 1$, $y = 0$, $x = 1$, and $x = 3$. The bounded region is rectangle.

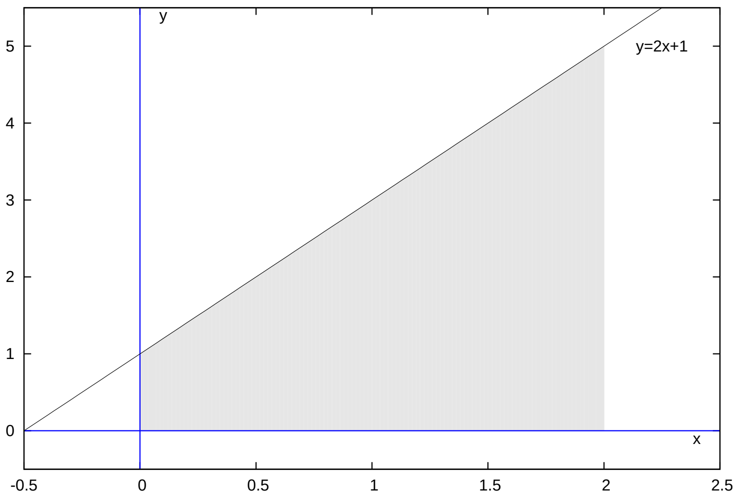


The area of the region is product of the base length and the height:

$$\int_1^3 1\,dx = (3 - 1) \cdot 1 = 2.$$

Example 2 Use geometry to evaluate $\int_0^2 (2x + 1)\,dx$.

To find the value of the integral, we can find the area of the region bounded by the graphs of $y = 2x + 1$, $y = 0$, $x = 0$, and $x = 2$. The bounded region is a trapezoid.



The area of the region is given by the average of the side lengths times the base length:

$$\int_0^2 (2x+1)\,dx = \left(\frac{1+5}{2}\right)\cdot 2 = 6.$$

Example 3 Use geometry to evaluate $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$.

The graph of $y = \sqrt{1-x^2}$ on $[-1, 1]$ is the upper half of the unit circle centered at $(0, 0)$. The area of one-half of the unit circle is $\pi/2$. Therefore

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}.$$

Other properties

Although proofs will be omitted, the following properties are pretty easy to believe if you use either interpretation of the definite integral—a limit of Riemann sums or an area under a curve.

Theorem 2 — Basic properties of definite integrals

Suppose f and g are integrable on $[a, b]$ and $[b, c]$.

- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ for any constant k
- $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Theorem 3 — Some inequalities involving definite integrals

Suppose f and g are integrable on $[a, b]$.

- If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.
- If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
- If $m \leq f(x) \leq M$ on $[a, b]$, then $m \cdot (b-a) \leq \int_a^b f(x)\,dx \leq M \cdot (b-a)$.

Theorem 4 — Even and odd functions in definite integrals

Suppose f is an even function (i.e. $f(-x) = f(x)$) and g is an odd function (i.e. $g(-x) = -g(x)$) on the interval $[-a, a]$.

- $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$
- $\int_{-a}^a g(x)\,dx = 0$

Example 4 Evaluate $\int_{-\pi}^{\pi} 8\sin x\,dx$.

$\sin x$ is an odd function. Using the results above we have

$$\int_{-\pi}^{\pi} 8\sin x\,dx = 8\int_{-\pi}^{\pi} \sin x\,dx = 8 \cdot 0 = 0.$$

Example 5 Suppose we know that $\int_1^3 3x^2\,dx = R$ and $\int_1^3 2x\,dx = S$. What is the value of $\int_3^1 (x^2 - x)\,dx$ in terms of R and S?

First notice that

$$3\int_1^3 x^2\,dx = R \quad\Longrightarrow\quad \int_3^1 x^2\,dx = -\frac{1}{3}R.$$

Similarly,

$$2\int_1^3 x\,dx = S \quad\Longrightarrow\quad \int_3^1 x\,dx = -\frac{1}{2}S.$$

It follows that

$$\int_3^1 (x^2 - x)\,dx = -\frac{1}{3}R + \frac{1}{2}S.$$

Example 6 Use the fact that $\int_0^b x^2\,dx = \frac{b^3}{3}$ to evaluate $\int_2^4 x^2\,dx$.

Solution omitted.

Average value

In order to find the average value of a collection of n numbers, we find the sum of the numbers and divide by n. But how would we compute the average value of a continuous function f on $[a, b]$? A plausible estimate might be obtained by finding the average of a sample of values of f at a number of points spread out in $[a, b]$. Let's work through the details:

Suppose f is defined on $[a, b]$. Partition $[a, b]$ into n subintervals of equal length, $\Delta x = \frac{b-a}{n}$. For each k from 1 to n, let c_k be a number chosen from the kth subinterval. The average value of f on $[a, b]$ is approximately

$$\text{Avg} \approx \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{[f(c_1) + f(c_2) + \cdots + f(c_n)]\Delta x}{b-a} = \frac{1}{b-a} \sum_{k=1}^{n} f(c_k)\Delta x.$$

Our approximation for the average value is nothing more than a Riemann sum. If f is integrable on $[a, b]$, we obtain the following definition by letting $n \to \infty$.

Definition of average value

Suppose f is Riemann integrable on $[a, b]$. The average value of f on $[a, b]$ is given by

$$\text{Avg} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example 7 Referring back to Example 3, find the average value of $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$.

$$\text{Avg} = \frac{1}{2} \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

It seems reasonable to believe that a continuous function on a interval must take on its average value at some point in the interval. This is indeed the case. The following "mean value" theorem for integrals is a consequence of the Intermediate Value Theorem and the third part of Theorem 3 (above).

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)\, dx = f(c) \cdot (b-a).$$

We will apply this mean value theorem in the next lecture.