Lecture 33: Integration by substitution

Objectives:

(33.1) Use substitution to evaluate indefinite integrals.

(33.2) Use substitution to evaluate definite integrals.

Substitution in indefinite integrals

Right now we have only one technique for finding an antiderivative—we reverse a familiar differentiation formula (i.e. we simply recognize and write the answer). While we have already reversed most of our basic differentiation rules, we have completely neglected some of the most difficult and important: the product rule, the quotient rule, and the chain rule. In this lecture we will focus on finding a practical way to integrate by reversing the chain rule. (The product and quotient rules will wait until Calculus II.)

The following integration formula is obtained by "undoing" the chain rule:

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C.$$

In order to use this formula, the integrand must have the correct form, with an outside function, an inside function, and the derivative of the inside function. Recognizing that we have the correct form is the hard part of using the formula. The key is a change-of-variable substitution involving the inside function.

For example, suppose we want to evaluate the integral $\int 2x\cos x^2\,dx$. If we notice that x^2 is an "inside" function and also that its derivative, $2x$, a factor of the integrand, we are motivated to try the following substitution:

$$u = x^2 \Longrightarrow \frac{du}{dx} = 2x \Longrightarrow du = 2x\,dx.$$

Upon substituting, we have

$$\int 2x\cos x^2\,dx = \int (\cos x^2)(2x\,dx) = \int \cos u\,du.$$

We've reduced the integral to one we immediately recognize. So we evaluate and resubstitute.

$$\int \cos u\,du = \sin u + C = \sin x^2 + C$$

Though difficult at first, we will come to find that this u-substitution technique is widely applicable and incredibly useful. When using it, there are a number of guidelines that we should keep in mind.

- When there is an inside function whose derivative is a factor of the integrand, we should try the substitution $u =$ inside function.
- A substitution should completely remove the old variable in favor of the new one. We cannot have mixed-variable integrals.
- An integral may require several substitutions.
- Sometimes the technique is used to clean up an integral, and this may be only one small step in the evaluation process.

Example 1 Use substitution to evaluate $\int (\sin x)^3 \cos x\,dx$.

The function $\sin x$ is an inside function and its derivative ($\cos x$) is a factor of the integrand.

$$u = \sin x \Longrightarrow \frac{du}{dx} = \cos x \Longrightarrow du = \cos x\,dx$$

This gives

$$\int (\sin x)^3 \cos x\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C.$$

Example 2 Use substitution to evaluate $\int x\sqrt{x^2+1}\,dx$.

We start by rewriting the integral: $\int x(x^2+1)^{1/2}\,dx$. The function x^2+1 is an inside function and its derivative, $2x$, is *almost* a factor of the integrand. As we will see, this almost is good enough.

$$u = x^2+1 \Longrightarrow \frac{du}{dx} = 2x \Longrightarrow du = 2x\,dx \Longrightarrow \frac{1}{2}\,du = x\,dx$$

This gives

$$\int x(x^2+1)^{1/2}\,dx = \int (x^2+1)^{1/2}(x\,dx) = \int u^{1/2}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{1/2}\,du.$$

We now evaluate the easy u-integral and resubstitute.

$$\frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C$$

Example 3 Use substitution to evaluate $\int 3\sec^2 7x\,dx$.

$$u = 7x \Longrightarrow \frac{du}{dx} = 7 \Longrightarrow du = 7\,dx \Longrightarrow \frac{1}{7}\,du = dx$$

This gives

$$\int 3\sec^2 7x\,dx = \int 3(\sec^2 7x)(dx) = \int 3(\sec^2 u)\left(\frac{1}{7}\,du\right) = \frac{3}{7}\int \sec^2 u\,du.$$

We now evaluate the easy u-integral and resubstitute.

$$\frac{3}{7}\int \sec^2 u\,du = \frac{3}{7}\tan u + C = \frac{3}{7}\tan 7x + C$$

Example 4 Use substitution to evaluate $\int \frac{6x^2}{(x^3+9)^2}\,dx$.

$$u = x^3+9 \Longrightarrow \frac{du}{dx} = 3x^2 \Longrightarrow du = 3x^2\,dx \Longrightarrow 2\,du = 6x^2\,dx$$

This gives

$$\int \frac{6x^2}{(x^3+9)^2}\,dx = \int \frac{6x^2\,dx}{(x^3+9)^2} = \int \frac{2\,du}{u^2} = \int 2u^{-2}\,du$$

We now evaluate the easy u-integral and resubstitute.

$$\int 2u^{-2}\,du = -2u^{-1} + C = -\frac{2}{u} + C = -\frac{2}{x^3+9} + C$$

Example 5 Use the substitution $u = 3x+2$ to evaluate $\int x\sqrt{3x+2}\,dx$.

$$u = 3x+2 \quad \Longrightarrow \quad \frac{1}{3}\,du = dx \quad \text{and} \quad x = \frac{u-2}{3}$$

This gives

$$\int \left(\frac{u-2}{3}\right) u^{1/2}\left(\frac{1}{3}\,du\right) = \frac{1}{9}\int (u^{3/2} - 2u^{1/2})\,du.$$

The remaining details are omitted.

Substitution in definite integrals

There are two different approaches to using substitutions in definite integrals. The most obvious approach is to completely evaluate the corresponding indefinite integral. Then evaluate at the bounds. The following example illustrates this approach.

Example 6 Use substitution to evaluate $\displaystyle\int_1^2 2x^2\sqrt{x^3+1}\,dx$.

We begin by completely evaluating the indefinite integral $\displaystyle\int 2x^2\sqrt{x^3+1}\,dx$.

$$u = x^3+1 \Longrightarrow du = 3x^2\,dx \Longrightarrow \frac{1}{3}\,du = x^2\,dx$$

This gives

$$\frac{2}{3}\int u^{1/2}\,du = \frac{4}{9}u^{3/2}+C = \frac{4}{9}(x^3+1)^{3/2}+C.$$

We now use the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\int_1^2 2x^2\sqrt{x^3+1}\,dx = \left[\frac{4}{9}(x^3+1)^{3/2}\right]_1^2 = \frac{4}{9}(9)^{3/2}-\frac{4}{9}(2)^{3/2} = \frac{4}{9}(27-2\sqrt{2}).$$

The other approach (the preferred approach!) is to carry out the substitution on both the integrand and the bounds. This approach is described in the following theorem.

Substitution in definite integrals

If the function $u = g(x)$ has a continuous derivative on $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 7 Evaluate $\displaystyle\int_0^{\pi/2}\cos\left(\frac{2x}{3}\right)\,dx$.

We let $u = 2x/3$.

$$u = \frac{2x}{3} \Longrightarrow du = \frac{2}{3}\,dx \Longrightarrow \frac{3}{2}\,du = dx$$

Now we change the x-bounds to the corresponding u-bounds:

$$x = 0 \Longrightarrow u = \frac{2(0)}{3} = 0 \qquad \text{and} \qquad u = \frac{\pi}{2} \Longrightarrow u = \frac{2(\pi/2)}{3} = \frac{\pi}{3}.$$

This gives

$$\int_0^{\pi/2}\cos\left(\frac{2x}{3}\right)\,dx = \frac{3}{2}\int_0^{\pi/3}\cos u\,du = \frac{3}{2}\sin u\Big|_0^{\pi/3} = \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{3}{2}(0) = \frac{3\sqrt{3}}{4}$$

Example 8 Evaluate $\displaystyle\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$.

The substitution $u = 1+\sqrt{x}$ gives

$$\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = 2\int_2^4 u^{-2}\,du.$$

The remaining details are omitted.