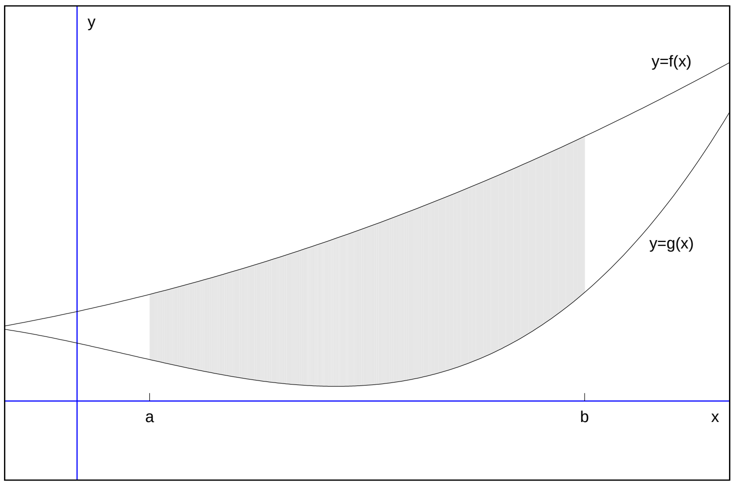
Lecture 34: Area between curves

Objectives:

(34.1) Find the area of a bounded region between the graphs of two functions.

Area between curves

Ever since its introduction, we have used the definite integral to compute area. In most cases, the region under consideration was bounded by the graph of a positive-valued function, the x-axis, and two vertical lines. In this lecture we consider regions between graphs such as that illustrated below.



If we interpret each of the integrals

$$\int_a^b f(x)\,dx \qquad \text{and} \qquad \int_a^b g(x)\,dx$$

as giving the area under each graph, respectively, then it is easy to see that the area of the shaded region is given by

$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx.$$

A nice thing about this new idea is that we no longer have to worry about our functions having positive values. If $f(x) \geq g(x)$, then we must have $f(x) - g(x) \geq 0$, regardless of the functions' individual values.

Theorem 1 - Area between curves

If f and g are continuous functions on $[a, b]$ with $f(x) \geq g(x)$, then the area between the graphs of f and g on $[a, b]$ is given by

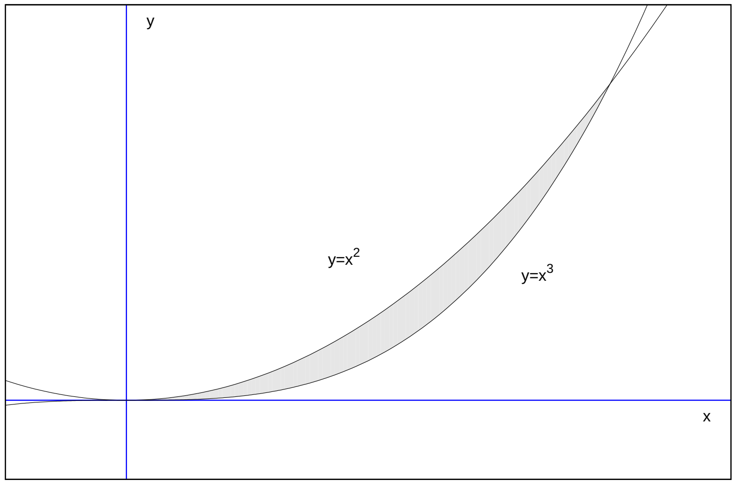
$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx.$$

Example 1 Find the area of the bounded region between the graphs of $y = x^2$ and $y = x^3$.

The graphs of $y = x^2$ and $y = x^3$ intersect when

$$x^2 - x^3 = x^2(1 - x) = 0 \quad \text{or} \quad x = 0,\, 1.$$

The bounded region extends from $x = 0$ to $x = 1$, and on this interval $x^2 \geq x^3$.

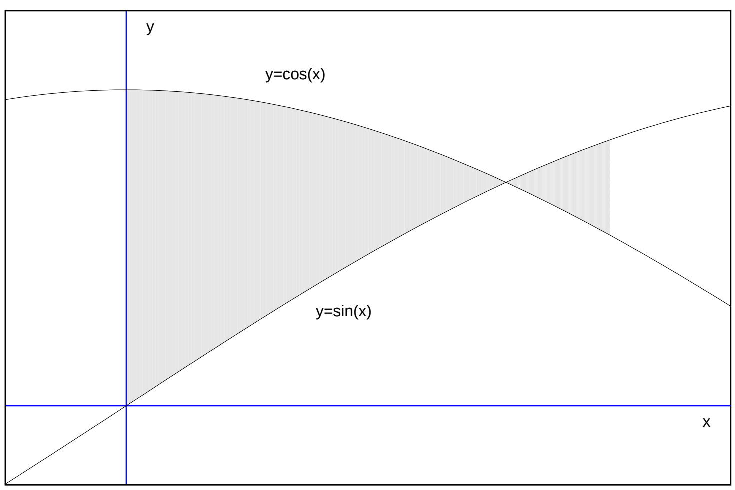


The area is given by

$$\text{Area} = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example 2 Find the area of the region between the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, 1]$.

The region is shown below.



The graphs of f and g intersect at $x = \pi/4 \approx 0.785$, and at this point the top graph and bottom graph switch. We must take this switch into account when computing the area:

$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^1 (\sin x - \cos x)\,dx = 2\sqrt{2} - 1 - \sin(1) - \cos(1) \approx 0.45,$$

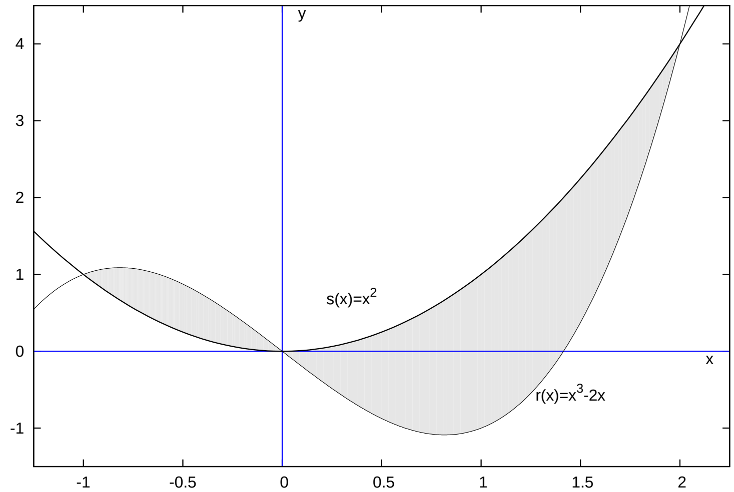
where the details have been omitted.

Although we could not have manually computed the area using a single integral, we could have described the area with one integral:

$$\int_0^1 |\sin x - \cos x|\,dx.$$

Wolfram Alpha could be used to evaluate this integral.

Example 3 Find the area of the region between the graphs of $r(x) = x^3 - 2x$ and $s(x) = x^2$.



The details are omitted, but we should find

$$\text{Area} = \int_{-1}^{0} [r(x) - s(x)]\, dx + \int_{0}^{2} [s(x) - r(x)]\, dx = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

Example 4 Find the positive number k such that the area of the region between the graphs of $y = kx$ and $y = x^2$ is 1.

Let k represent some unknown positive number. The graphs of $y = kx$ and $y = x^2$ intersect when

$$kx - x^2 = x(k - x) = 0 \quad \text{or} \quad x = 0,\, k.$$

On the interval $[0, k]$, $kx \geq x^2$. Therefore the area of the region between the graphs is given by

$$\text{Area} = \int_{0}^{k} (kx - x^2)\, dx = \left[\frac{k}{2}x^2 - \frac{1}{3}x^3\right]_0^k = \frac{k^3}{2} - \frac{k^3}{3} = \frac{k^3}{6}.$$

It follows that

$$\text{Area} = 1 \quad \Longrightarrow \quad k = \sqrt[3]{6}.$$