Lecture 35: The trapezoid rule

Objectives:

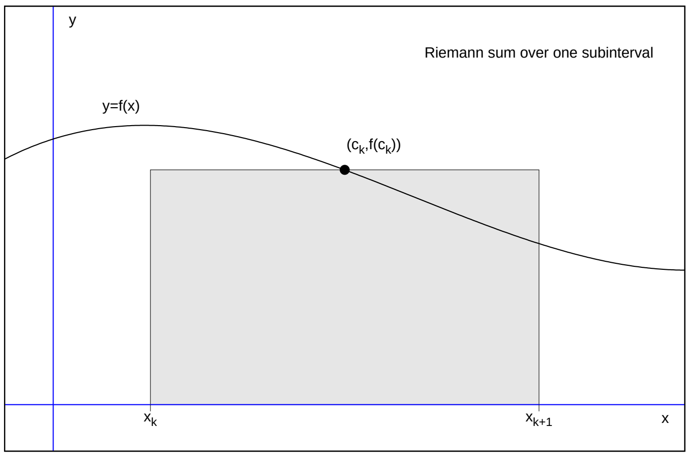
(35.1) Use the trapezoid rule to approximate the value of a definite integral.

(35.2) Find an upper bound on the error made in using the trapezoid rule to approximate a definite integral.

The trapezoid rule

Our introduction to the definite integral began with approximations based on Riemann sums. Theoretically Riemann sums provide a very nice approach to approximating integrals, but in practice, we can do better.

When we use a Riemann sum to approximate an integral, we are actually approximating the integrand with a constant function (on each subinterval) and then exactly integrating the constant function. This is illustrated below.



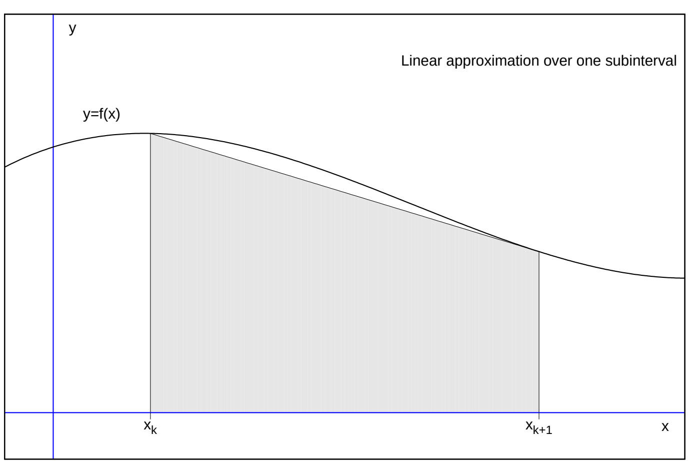
On the interval $[x_k, x_{k+1}]$, we approximate $y = f(x)$ with the constant function $y = f(c_k)$. This gives

$$\int_{x_k}^{x_{k+1}} f(x)\,dx \approx \int_{x_k}^{x_{k+1}} f(c_k)\,dx = f(c_k)\int_{x_k}^{x_{k+1}} 1\,dx = f(c_k)(x_{k+1} - x_k) = f(c_k)\Delta x_k.$$

By repeating this over all subintervals, we obtain a typical Riemann sum.

It is natural to wonder how we might improve our approximation while keeping the subintervals fixed. One approach is to find a better approximation for the integrand itself. If we are not satisfied with a constant function, let's try a linear function. The linear function passing through the points $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$ is given by

$$y = f(x_k) + \frac{f(x_{k+1}) - f(x_k)}{\Delta x_k}(x - x_k).$$



On the interval $[x_k, x_{k+1}]$, we approximate $f(x)$ with the linear function given above. This gives (omitting some details)

$$\int_{x_k}^{x_{k+1}} f(x)\,dx \approx \int_{x_k}^{x_{k+1}} \left(f(x_k) + \frac{f(x_{k+1}) - f(x_k)}{\Delta x_k}(x - x_k) \right)\,dx = \frac{\Delta x_k}{2}(f(x_k) + f(x_{k+1})).$$

Because the integral above gives the area of a trapezoid, this method of approximation is called the *trapezoid rule*. If we insist that all subintervals have the same width and we use the trapezoid rule over each one, then we obtain the following approximation method.

Trapezoid rule over n subintervals

Suppose f is integrable on $[a, b]$. Partition $[a, b]$ into n subintervals of equal length $h = \dfrac{b-a}{n}$:

$$a = x_0 < x_1 = a + h < x_2 = a + 2h < \cdots < x_k = a + kh < \cdots < x_n = b.$$

The trapezoid rule approximation for $\displaystyle\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Example 1 Use the trapezoid rule over four subintervals to approximate $\displaystyle\int_0^1 x^2\,dx$. Compare the approximation with the exact value.

Using four subintervals, we have $h = \dfrac{1-0}{4} = 0.25$ and therefore

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1.$$

The trapezoid rule approximation is

$$\int_0^1 x^2\,dx \approx \frac{0.25}{2}\left[0^2 + 2(0.25)^2 + 2(0.5)^2 + 2(0.75)^2 + 1^2\right] = 0.34375.$$

The exact value of the integral is given by

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} = 0.333333\ldots.$$

Example 2 Use the trapezoid rule over eight subintervals to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

Since the function $f(x) = \dfrac{1}{x}$ is continuous on $[1, 2]$, this integral definitely exists. However, we have not studied a function whose derivative is $1/x$. Therefore the Fundamental Theorem of Calculus cannot be applied to evaluate the integral. The trapezoid rule is particularly appropriate for this problem.

Using $n = 8$ we have $h = \dfrac{2-1}{8} = 0.125$. So our partition is

$$x_0 = 1, \quad x_1 = 1.125, \quad x_2 = 1.25, \quad x_3 = 1.375, \quad x_4 = 1.5,$$

$$x_5 = 1.625, \quad x_6 = 1.75, \quad x_7 = 1.875, \quad x_8 = 2.$$

The trapezoid rule approximation is

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.125}{2}\left[\frac{1}{1} + \frac{2}{1.125} + \frac{2}{1.25} + \frac{2}{1.375} + \frac{2}{1.5} + \frac{2}{1.625} + \frac{2}{1.75} + \frac{2}{1.875} + \frac{1}{2}\right] \approx 0.69412185.$$

Example 3 Some values of the function f are given in the table below. Use the trapezoid rule to approximate $\int_0^2 f(x)\,dx$.

x	0	0.4	0.8	1.2	1.6	2
$f(x)$	2.3	1.8	1.6	1.3	0.8	1.7

The x-values in the table are all separated by $h = 0.4$. Therefore it makes sense to use this h and all the data values. The trapezoid rule approximation is given by

$$\int_0^2 f(x)\,dx \approx \frac{0.4}{2}\,[2.3 + 2(1.8) + 2(1.6) + 2(1.3) + 2(0.8) + 1.7] = 3.0.$$

Trapezoid rule in a CAS

A Maxima program for using the trapezoid rule is given below. The user input is in the first few lines, and the required modifications should be obvious.

```
/* Trapezoid Rule */
f: 1/x   $
a: 1   $
b: 2   $
N: 8   $

h: float( b - a ) / N$
sum: ( ev( f, numer, x=a ) + ev( f, numer, x=b ) ) / 2.0$
t: float( a )$
for i: 1 thru N-1 do
(  t: t + h,
   sum: sum + ev( f, numer, x=t )
)$
print( "Trapezoid rule approximation is: ", h * sum )$
```

Error in the trapezoid rule

When using an approximation technique, it is important to be able to judge the accuracy of an approximation. The following theorem gives an upper bound on the error made in using the trapezoid rule to approximate a definite integral.

Theorem 1 — Error in the trapezoid rule

Suppose f has a continuous second derivative on $[a, b]$ and let T be the trapezoid rule approximation

$$T = \frac{b-a}{2n}\,[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] \approx \int_a^b f(x)\,dx.$$

The error made in using T to approximate the integral satisfies the inequality

$$\left| T - \int_a^b f(x)\,dx \right| \le \frac{(b-a)^3}{12n^2}\,\max\{|f''(x)| : a \le x \le b\}.$$

It should be clear from our derivation of the trapezoid rule as well as its error estimate that the trapezoid rule is exact for polynomials of degree one. In other words, if f is a polynomial of degree 0 or 1, then

$$T = \int_a^b f(x)\,dx$$

for any interval $[a, b]$ and for any number of subintervals (even $n = 1$).

Example 4 Use Theorem 1 to estimate the error made in the approximation of Example 2.

Referring back to Example 2, we have $a = 1$, $b = 2$, $n = 8$ and $f(x) = \frac{1}{x}$. It follows that $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$. The maximum value of $|f''(x)|$ on $[1, 2]$ occurs at $x = 1$ where $f''(1) = 2$. According to the theorem, the error made in the approximation is less than

$$\frac{(2-1)^3}{12(8)^2} \cdot 2 = \frac{2}{768} \approx 0.0026.$$

The approximation found in Example 2 differs from the exact value of he definite integral by less than 0.0026.

Example 5 Determine the number of subintervals required to approximate $\int_0^2 \cos x \, dx$ with an error less than 10^{-6}. Then use the Maxima program to find the approximation using that number of subintervals.

Referring to the theorem above with $a = 0$, $b = 2$, $f(x) = \cos x$, and $f''(x) = -\cos x$, we must find n so that

$$\frac{(b-a)^3}{12n^2} \max\{|f''(x)| : a \leq x \leq b\} = \frac{2}{3n^2} \max\{|-\cos x| : 0 \leq x \leq 2\} < 10^{-6}.$$

Since $-1 \leq -\cos x \leq 1$ for all x, we may use 1 as an upper bound on $|-\cos x|$. So we must find n satisfying

$$\frac{2}{3n^2} < 10^{-6}$$

or

$$\frac{3n^2}{2} > 10^6.$$

It follows that

$$n > \sqrt{\frac{2 \times 10^6}{3}} \approx 816.5.$$

We should use $n = 817$ to be certain to achieve the required accuracy. (That's a lot of subintervals!)

Using $n = 817$ in the Maxima program given above we get

$$\int_0^2 \cos x \, dx \approx 0.909296972737.$$

An improved trapezoid rule (optional)

If we are willing to evaluate $f'(x)$ at $x = a$ and $x = b$, we can make a significant improvement to a trapezoid rule approximation.

Theorem 2 — Correction for the trapezoid rule

Suppose f has a continuous third derivative on $[a, b]$ and let IT be the improved trapezoid rule approximation

$$IT = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] + \frac{(b-a)^2}{12n^2}\left[f'(a) - f'(b)\right] \approx \int_a^b f(x)\, dx.$$

The error made in using IT to approximate the integral satisfies the inequality

$$\left| IT - \int_a^b f(x)\, dx \right| \leq \frac{(b-a)^4}{240n^3}(M - m),$$

where m and M are the minimum and maximum values of $|f'''(x)|$ on $[a, b]$.

Notice that the improved trapezoid rule approximation can be written

$$IT = T + \frac{h^2}{12}\,[f'(a) - f'(b)],$$

where T is the original trapezoid rule approximation.

Example 6 Apply the improvement to the trapezoid rule approximation of Example 2.

In Example 2, we had $a = 1$, $b = 2$, $h = 0.125$, and $f(x) = \dfrac{1}{x}$. It follows that $f'(x) = -\dfrac{1}{x^2}$. Therefore

$$IT = T + \frac{(0.125)^2}{12}\,[-1 + 0.25] = 0.6931453.$$

Example 7 Use Theorem 2 to estimate the error made in the approximation of Example 6.

We have $a = 1$, $b = 2$, $n = 8$, and $f'''(x) = -\dfrac{6}{x^4}$. On the interval $[1, 2]$, the minimum value of $|f'''(x)|$ is $6/16$, which occurs at $x = 2$. The maximum value is 6, which occurs at $x = 1$. An upper bound on the error is given by

$$\frac{(2-1)^4}{240(8)^3} \cdot \left(6 - \frac{6}{16}\right) \approx 0.000046.$$

The exact value of the definite integral differs from the value of IT by less than 0.000046.