Lecture 4: Finding limits analytically—Limit laws and substitution

Objectives:

(4.1) Use limit laws to evaluate limits.

(4.2) Use substitution to evaluate limits.

Limit laws

Tables and graphs are useful for estimating limits. However, without additional information we can NEVER have complete confidence in a result obtained from a table or graph. We need a better approach. The ideas presented in this lecture start us down that path. These ideas can be proved rigorously, but we need a formal definition of the limit concept before we can give the proofs. That will come later. For now, let's take a look at a handful of rules for evaluating limits.

Theorem 1 — Limit laws
Suppose k and r are real numbers. Also suppose that f and g are functions such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.

1. $\lim_{x\to c} k = k$
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
4. $\lim_{x\to c}[f(x)g(x)] = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$
5. Special case of (4): $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$
6. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$
7. $\lim_{x\to c}[f(x)]^r = \left[\lim_{x\to c} f(x)\right]^r$, provided $[f(x)]^r$ is defined on an open interval containing c
8. Special case of (7): $\lim_{x\to c} x^r = c^r$, provided x^r is defined on an open interval containing c

In contrast to graphical and numerical methods, the limit laws can be used to formally evaluate limits. When applicable and when used properly, the limit laws do not give approximations, they give the actual limits.

Example 1 Use the limit laws to evaluate $\lim_{x\to 2}(3x^2 + 4x)$.

Using laws 2, 3, 5, and 8, we have

$$\lim_{x\to 2}(3x^2 + 4x) = 3\left(\lim_{x\to 2} x^2\right) + 4\left(\lim_{x\to 2} x\right) = 3(2^2) + 4(2) = 20.$$

Example 2 Use the limit laws to evaluate $\lim_{x\to 4} \dfrac{6+\sqrt{x}}{x^2(3x+1)}$.

Using practically all of the laws, we have

$$\lim_{x\to 4} \frac{6+\sqrt{x}}{x^2(3x+1)} = \frac{\lim_{x\to 4} 6 + \sqrt{\lim_{x\to 4} x}}{\left(\lim_{x\to 4} x^2\right)\left(3\lim_{x\to 4} x + \lim_{x\to 4} 1\right)} = \frac{6+\sqrt{4}}{(16)(3\cdot 4+1)} = \frac{8}{(16)(13)} = \frac{1}{26}.$$

Example 3 Suppose $\lim_{x\to 5} f(x) = 7$ and $\lim_{x\to 5} g(x) = 2$. Use the limit laws to evaluate $\lim_{x\to 5} \dfrac{(g(x))^3}{f(x)+1}$.

$$\lim_{x\to 5} \frac{(g(x))^3}{f(x)+1} = \frac{\left(\lim_{x\to 5} g(x)\right)^3}{\lim_{x\to 5} f(x) + \lim_{x\to 5} 1} = \frac{2^3}{7+1} = 1.$$

Example 4 Carefully explain why the limit laws cannot be used to evaluate the following limits.

(a) $\lim_{x\to -2} \dfrac{x^2-4}{x+2}$ (b) $\lim_{x\to 1} \sqrt{x-1}$

Solutions omitted.

The limit laws could not be applied in Example 4. Nevertheless, the limit in Example 4a actually exists. Students are commonly mislead into believing that if the limits laws do not apply, then the limit must not exist. Be careful to use the limit laws only when they apply, and do not draw unfounded conclusions from them. **When evaluating limits, if the limit laws cannot be used, we must simply use some other method to determine the limit.**

Some consequences of the limit laws

The next theorem describes two rather obvious consequences of the limit laws.

Theorem 2 — Limits of polynomials and rational functions
Suppose P is a polynomial and R is a rational function (i.e. a quotient of two polynomials).

1. $\lim_{x\to c} P(x) = P(c)$ for any real number c.
2. $\lim_{x\to c} R(x) = R(c)$ for any real number c, provided $R(c)$ exists.

This is a very useful result. It tells us that we may evaluate limits of polynomials or rational functions simply by substituting the limit point into the expression, provided that we do not attempt to divide by zero.

Example 5 Evaluate the limit $\lim_{x\to 3}(5x^2 - 7x + 6)$.

We are looking for the limit of a polynomial function. The limit can be obtained by evaluating the polynomial at the limit point.

$$\lim_{x\to 3}(5x^2 - 7x + 6) = 5(3)^2 - 7(3) + 6 = 30$$

Example 6 Evaluate the limit $\lim_{x\to 0} \dfrac{x^3+7x-9}{5x^2-x+3}$.

We are looking for the limit of a rational function. The limit can be obtained by evaluating the function at the limit point.

$$\lim_{x\to 0} \frac{x^3+7x-9}{5x^2-x+3} = \frac{-9}{3} = -3$$

Example 7 Let $f(x) = \dfrac{x-9}{\sqrt{x}-3}$. Explain why the theorem cannot be used to evaluate $\lim_{x\to 9} f(x)$.

$f(9)$ is not defined. The theorem tells us absolutely nothing about the limit. In order to find the limit, we would need to use a different approach.

Some more limit laws

These last two theorems give us a great deal more to work with.

Theorem 3 — Limits of the trig functions
Let trig(x) represent any one of the six basic trigonometric functions. Then

$$\lim_{x\to c} \operatorname{trig}(x) = \operatorname{trig}(c),$$

provided trig(x) is defined at $x = c$.

Theorem 4 — Limits of compositions of functions
If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L).$$

The theorems given here provide the motivation for our first strategy for evaluating limits analytically. We will call this the substitution technique:

> **Suppose f is defined by a single elementary expression on an open interval containing c, except possibly at c. Then the limit of f at c can be obtained by substituting the limit point c into the expression for f, unless the expression is not defined at c. If the expression is not defined at the limit point, we simply cannot use this technique, but the limit may or may not exist.**

Example 8 Evaluate the limit $\lim_{x\to\pi} (x^2 \sin x + 3x\cos x)$.

Substitute $x = \pi$.

$$\lim_{x\to\pi} (x^2 \sin x + 3x\cos x) = \pi^2 \sin\pi + 3\pi\cos\pi = (\pi^2)(0) + 3\pi(-1) = -3\pi$$

Example 9 Evaluate the limit $\lim_{x\to 2} \tan\left(\pi\sqrt{\frac{x}{32}}\right)$.

Substitute $x = 2$.

$$\lim_{x\to 2} \tan\left(\pi\sqrt{\frac{x}{32}}\right) = \tan\frac{\pi}{4} = 1$$

Example 10 Explain why substitution cannot be used to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution omitted.