

Lecture 7: Continuity

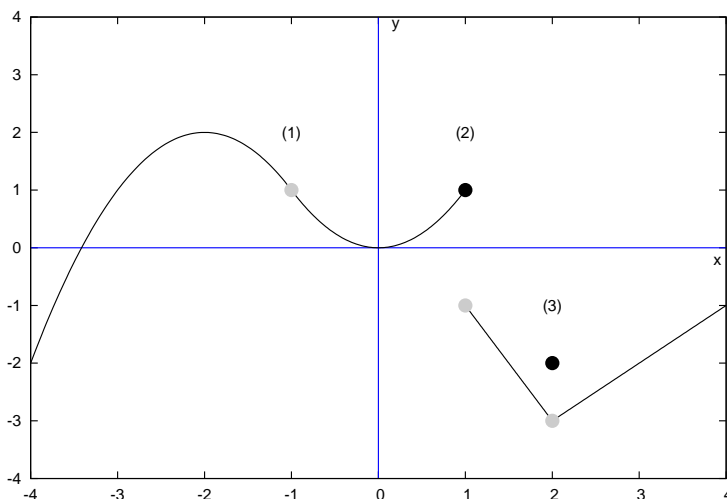
Objectives:

- (7.1) Use the definition of continuity to determine if a function is continuous at a point.
- (7.2) Use the properties of continuity to determine if a function is continuous at a point.
- (7.3) Classify discontinuities.
- (7.4) Apply the Intermediate Value Theorem.

Definition and properties of continuity

It is not uncommon in algebra classes to think about the continuity of a function in terms of its graph. Specifically, we learned that a function is continuous if we can sketch its graph without picking up our pencils. It is worthwhile to change our point of view and disassociate continuity and graphing. The limit concept provides the tool we need to do so.

If we think about a point of discontinuity as being caused by a lifting of the pencil, it is intuitively clear that the graph has a corresponding break. The graph below illustrates the possible types of breaks.



We see that breaks occur at points where (1) the function is not defined, (2) the limit does not exist, or (3) the limit does not equal the function value. If a function is to be continuous at a point, we must simply require that none of these “problems” arise. *A function is continuous at $x = c$ if the value the function is approaching as x approaches c is the value that the function actually attains at $x = c$.*

Definition of continuity

The function f is continuous at an interior point c of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

If c is an interval endpoint, then the limit is replaced by a corresponding one-sided limit.

According to this definition, **three conditions must be met** if f is to be continuous at c :

1. f must be defined at c
2. The limit of f at c must exist
3. The limit of f at c must equal the value of f at c

Example 1 Determine a number k so that f is continuous at $x = 0$. Use the definition of continuity.

$$f(x) = \begin{cases} k + 3 \sin x, & x \leq 0 \\ x + 1, & x > 0 \end{cases}$$

In order for f to be continuous at $x = 0$, we must ensure that $\lim_{x \rightarrow 0} f(x) = f(0)$. First, regardless of the value of k , the function f is defined at $x = 0$ and $f(0) = k$. (If f was not defined at $x = 0$, it could not possibly be continuous there.) In order for the limit to exist, the left and right limits must be equal. So let's evaluate the limits and set them equal:

$$\lim_{x \rightarrow 0^-} f(x) = k + 3 \sin 0 = k \qquad \lim_{x \rightarrow 0^+} f(x) = 0 + 1 = 1$$

We must have $k = 1$.

Example 2 Let $f(x) = 3x^2 - x + 7$.

1. Show that f is continuous at $x = 1$.

f is defined at $x = 1$ and $f(1) = 9$. The two-sided limit of f at $x = 1$ can be determined by direct substitution:

$$\lim_{x \rightarrow 1} f(x) = 9.$$

Since $\lim_{x \rightarrow 1} f(x) = f(1)$, the function f is continuous at $x = 1$.

2. At which other points is f continuous?

f is a polynomial function. It is defined for all real numbers and its limit at any point can be determined by direct substitution. Therefore

$$\lim_{x \rightarrow c} f(x) = f(c)$$

for any number c . f is continuous everywhere.

According to our definition of continuity, a function is continuous at any point at which the limit can be determined by direct substitution. The next theorem follows directly from the results of Lecture 4.

Theorem 1 — Properties of continuity

1. Polynomial functions are continuous everywhere.
2. Rational functions are continuous wherever they are defined.
3. The six basic trigonometric functions are continuous wherever they are defined.
4. Radical functions are continuous wherever they are defined.
5. Sums, differences, products, quotients, and compositions of continuous functions are continuous wherever they are defined.

With the help of theorem 1, we can easily analyze the continuity of elementary functions.

Example 3 Use theorem 1 to discuss the continuity of $h(x) = \frac{\sqrt{x}}{\cos x}$.

$y = \sqrt{x}$ and $y = \cos x$ are continuous wherever they are defined. Therefore the quotient is continuous wherever it is defined. h is continuous at each point of the set

$$\{x : x \geq 0 \text{ and } \cos x \neq 0\}.$$

Classifying discontinuities

Because discontinuities occur for a variety of reasons, it is often convenient to give a descriptive name to each particular type of discontinuity.

Jump Discontinuity — If the limits from the left and right exist at $x = c$, but are not equal, the corresponding discontinuity is called a **jump discontinuity**. (See Discontinuity (2) in the figure above.)

Infinite Discontinuity — If the limit at $x = c$ does not exist because the function values grow without bound as x approaches c (from either side), the corresponding discontinuity is called an **infinite discontinuity**. (For example, $f(x) = 1/x$ has an infinite discontinuity at $x = 0$.)

Removable Discontinuity — If the function is discontinuous at $x = c$, but the limit exists at $x = c$, the discontinuity is said to be **removable**. (See Discontinuities (1) and (3) in the figure above.)

Nonremovable Discontinuity — If the function is discontinuous at $x = c$ and the limit at $x = c$ does not exist, the discontinuity is said to be **nonremovable**. (See Discontinuity (2) in the figure above.)

Example 4 Consider the function $f(x) = \frac{x^2 - 1}{(x - 2)(x - 1)}$.

1. Find and classify the discontinuities of f .

Since f is a rational function it is continuous wherever it is defined. Therefore the only discontinuities occur at $x = 2$ and $x = 1$, i.e. those points at which f is not defined. Let's check the limits. At $x = 2$, direct substitution yields the form $3/0$. This is not an indeterminate form, and it indicates that the function values are unbounded as x approaches 2. The discontinuity at $x = 2$ is a nonremovable, infinite discontinuity. At $x = 1$, the limit exists:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{(x+1)}{(x-2)} = \lim_{x \rightarrow 1} \frac{x+1}{x-2} = -2.$$

The discontinuity at $x = 1$ is a removable discontinuity.

2. Define f at $x = 1$ so that the new function is continuous at $x = 1$.

If we give f the value of -2 at $x = 1$, then f will be continuous at that point since we will have

$$\lim_{x \rightarrow 1} f(x) = -2 = f(1).$$

To do so, we redefine f as a piecewise function:

$$f(x) = \begin{cases} \frac{x^2 - 1}{(x - 2)(x - 1)}, & x \neq 1, 2 \\ -2, & x = 1 \end{cases}$$

f is still discontinuous at $x = 2$, but there is nothing we can do about that. (The discontinuity at $x = 2$ is nonremovable.)

Example 5 Give an example of a function with nonremovable discontinuities at $x = 0$ and $x = 1$ and with removable discontinuities at $x = 2$ and $x = 3$.

Solution omitted.

The Intermediate Value Theorem

Of course there are lots of reasons why continuous functions are so nice. One is that limits of continuous functions can always be determined by direct substitution. Another reason is that continuous functions must take on all values between any two function values. For example, if f is a continuous function with $f(0) = 5$ and $f(2) = 10$, then f must attain all values between 5 and 10 on the interval from 0 to 2. This idea is easy to believe if we experiment by sketching the graphs of some continuous functions. While it is not particularly easy to prove, the idea gives rise to the Intermediate Value Theorem.

Intermediate Value Theorem

Suppose f is continuous on the interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there must exist a number c in $[a, b]$ such that $f(c) = k$.

Example 6 Let $f(x) = x^3 - 5x + 3$.

1. Use the Intermediate Value Theorem to argue that $f(x) = 0$ has a solution somewhere in the interval $[0, 1]$.

f is continuous on $[0, 1]$. Therefore on the interval $[0, 1]$, f must take on every value between $f(0) = 3$ and $f(1) = -1$. In particular, f must take on the value 0 somewhere in $[0, 1]$.

2. Further argue that the solution lies somewhere between 0.5 and 1.

The Intermediate Value Theorem applies on $[0.5, 1]$. Since $f(0.5) = 0.625$ and $f(1) = -1$, $f(x) = 0$ must occur somewhere between $x = 0.5$ and $x = 1$.

3. Further argue that the solution lies somewhere between 0.5 and 0.75.

Solution omitted.