Definition of concavity

Suppose f is differentiable on (a, b).

- If f' is increasing on (a, b), then the graph of f is concave up on (a, b).

- If f' is decreasing on (a, b), then the graph of f is concave down on (a, b).

Theorem — Test for concavity

Suppose f is a function such that f'' exists on (a, b).

- If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave up on (a, b).

- If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave down on (a, b).

Definition of Hergert number

An interior point in the domain of f at which $f''(x) = 0$ or $f''(x)$ does not exist is called a Hergert number.

Hergert numbers are domain points at which a graph's concavity might change.

Definition of inflection point

Suppose that f is continuous on the interval (a, b) containing the point c. If the graph of f has a tangent line at $(c, f(c))$ and the concavity of the graph changes at that point, then $(c, f(c))$ is called a point of inflection of the graph of f.

Warning: The existence of the tangent line is not always required in the definition of inflection point.

Theorem — Inflection points and Hergert numbers

If $(c, f(c))$ is a point of inflection of the graph of f, then c is a Hergert number of f.

Finding intervals on which the graph of f is concave up/down

Suppose that f is continuous on the interval I. Also suppose that f is twice differentiable inside I, except possibly at some isolated points. We use the following steps to find open intervals on which the graph of f is concave up/down.

1. Determine domain endpoints (if any) and all points at which $f''(x)$ is zero or not defined. (This list will include all Hergert numbers of f.)

2. Draw a number line and mark the points found in step 1.

3. Determine the sign of f'' on each interval along your number line.

4. List open intervals on which the graph of f is concave up/down.

5. Identify all inflection points.

Theorem — Second derivative test for relative extrema

Suppose that f is twice differentiable on an open interval containing c and that $f'(c) = 0$.

- If $f''(c) > 0$, then the graph of f is concave up at c and $f(c)$ is a relative minimum.
- If $f''(c) < 0$, then the graph of f is concave down at c and $f(c)$ is a relative maximum.
- If $f''(c) = 0$, then this test fails to be useful and another test must be applied.