## Trapezoid rule over $n$ subintervals

Suppose $f$ is integrable on $[a, b]$. Partition $[a, b]$ into $n$ subintervals of equal length $h = \dfrac{b-a}{n}$:

$a = x_0 < x_1 = a{+}h < x_2 = a{+}2h < \cdots < x_k = a{+}kh < \cdots < x_n = b.$

The trapezoid rule approximation for $\int_a^b f(x)\,dx$ is given by

$$\int_a^b f(x)\,dx \approx T,$$

where

$$T = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

## Theorem — Error in the trapezoid rule

Suppose $f$ has a continuous second derivative on $[a, b]$ and let $T$ be a trapezoid rule approximation for $\int_a^b f(x)\,dx$. The error in the approximation satisfies the inequality

$$\left|T - \int_a^b f(x)\,dx\right| \leq \frac{(b-a)^3}{12n^2} \max\{|f''(x)| : a \leq x \leq b\}.$$

## Theorem — Correction for the trapezoid rule

Suppose $f$ has a continuous third derivative on $[a, b]$ and let $T$ be a trapezoid rule approximation for $\int_a^b f(x)\,dx$. An improved trapezoid rule approximation is given by

$$IT = T + \frac{(b-a)^2}{12n^2}[f'(a) - f'(b)].$$

Furthermore, the error in the approximation satisfies the inequality

$$\left| IT - \int_a^b f(x)\,dx \right| \leq \frac{(b-a)^4}{240n^3}(M - m),$$

where $m$ and $M$ are the minimum and maximum values of $|f'''(x)|$ on $[a, b]$.