Lecture 9: Formal definition of limit

Objectives:

(9.1) Use the formal definition of limit to prove certain limits.

(9.2) Find a δ corresponding to a particular ϵ by using analytical, numerical, or graphical techniques.

Formalizing the limit concept

Our informal definition of the limit concepts says:

> Suppose the function f is defined on an open interval containing the number c, but f need not be defined at c. If $f(x)$ can be made arbitrarily close to the number L by choosing x sufficiently close to, but different from, c then we say $\lim_{x\to c} f(x) = L$.

The way it sits right now, this definition is not rigorous enough for us to be able to prove the limit laws or the other limit theorems that we have been using. We need a more formal definition. Specifically, we need to carefully define what is meant by "arbitrarily close" and "sufficiently close."

Recall that we use $|a-b|$ to measure the distance along the number line from a to b. To say that a and b can be made arbitrarily close simply means that $|a-b|$ can be made as small as we'd like (except equal to zero). For example, $0.999\ldots 9$ can be made arbitrarily close to to 1 by appending more 9's. The formal definition of limit will follow directly from our informal definition once we quantify the idea of closeness.

Formal Definition of Limit
Suppose f is defined on an open interval containing c, except possibly at c. The limit of f at c is L if, for any positive number ϵ, there exists a number δ, such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

The number ϵ measures the closeness of $f(x)$ and L, while δ measures the closeness of x and c. To use the formal definition to prove that a limit exists, we must show that a δ exists for any positive ϵ.

Example 1 We already know that $\lim_{x\to 2}(4x+2) = 10$.

1. Find a δ corresponding to $\epsilon = 0.1$.

   We need to find a number δ so that $0 < |x-2| < \delta$ implies that $|(4x+2) - 10| < 0.1$. To do so, notice that

   $$|4x + 2 - 10| < 0.1 \iff |4x - 8| < 0.1 \iff 4|x - 2| < 0.1 \iff |x - 2| < 0.1/4.$$

   Therefore, $|(4x+2) - 10| < 0.1$ whenever $|x-2| < 0.1/4 = 0.025$. The number corresponding to $\epsilon = 0.1$ is $\delta = 0.025$.

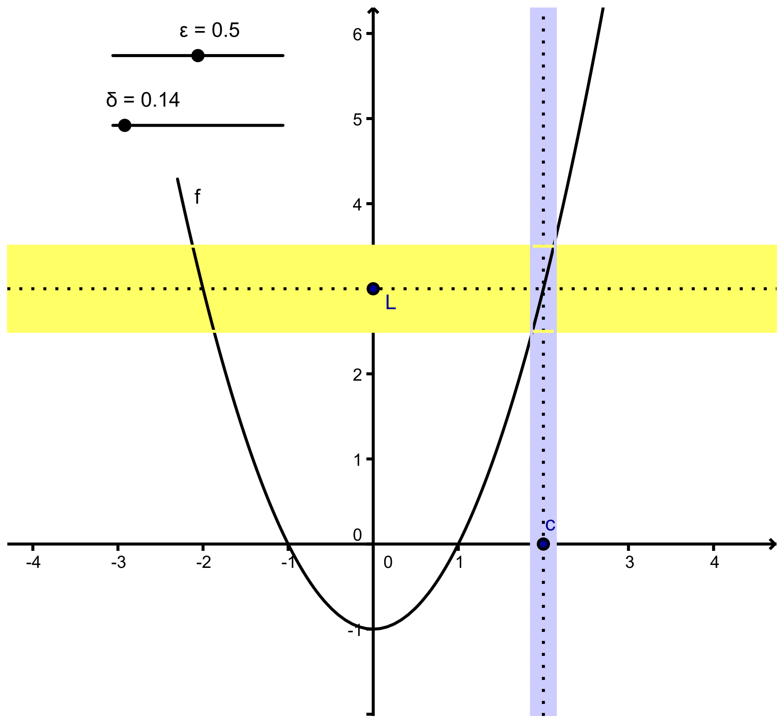
2. Find a δ corresponding to an arbitrary ϵ.

   The argument above continues to hold if we replace 0.1 with ϵ. The number corresponding to ϵ is $\delta = \epsilon/4$.

Example 2 We know that $\lim_{x\to -1}(5x+7) = 2$. Use the formal definition to prove it.

$\delta = \epsilon/5$. *Details omitted.*

According to the formal definition, $\lim_{x\to c} f(x) = L$ if, for any positive ϵ, the values of $f(x)$ are within ϵ units of L, whenever x is within δ units of c. It can be helpful to think about this concept graphically.



The horizontal band shows all y-values within ϵ of L. If the limit exists, then for any ϵ, there is a vertical band at $x = c$ of radius δ that contains the graph and is entirely inside the horizontal band.

Example 3 Experiment with the GeoGebra applet at `http://stevekifowit.com/geo_apps`. Consider the limit of $f(x) = x^2 + 4x + 3$ at $x = 0$. Estimate a δ that corresponds to $\epsilon = 0.75$.

$\delta \approx 0.2$

More difficult examples (optional)

Example 4 Use the formal definition to prove that $\lim_{x \to 2} (x^2 - x) = 2$.

First, since we are taking the limit at $x = 2$, there is no harm in restricting our attention to x-values satisfying $1 \le x \le 3$. Now let ϵ be any positive number. We are looking for a δ such that $|x^2 - x - 2| < \epsilon$. This is equivalent to

$$|(x-2)(x+1)| < \epsilon \quad \text{or} \quad |x-2||x+1| < \epsilon$$

and since $1 \le x \le 3$, we must have $2 \le |x+1| \le 4$.

We've almost got it! Now if we choose $\delta = \epsilon/4$, then

$$|x-2| < \delta \Longrightarrow |x-2| < \epsilon/4 \Longrightarrow \underbrace{4|x-2| < \epsilon \Longrightarrow |x-2||x+1| < \epsilon}_{\text{since } |x+1|<4} \Longrightarrow |x^2 - x - 2| < \epsilon.$$

The number corresponding to ϵ is $\delta = \epsilon/4$.

Example 5 Use the formal definition to prove the following limit law: If $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} k\,f(x) = kL$ for any real number k.

First, if $k = 0$, the result is clearly true. Now, assume $k \ne 0$ and let ϵ be an arbitrary positive number. Notice that $\epsilon/|k|$ is a positive number, and since the limit of f at c is L, there exists a δ corresponding to the number $\epsilon/|k|$ such that $|f(x) - L| < \epsilon/|k|$ whenever $0 < |x - c| < \delta$. So now if $0 < |x - c| < \delta$, then

$$|kf(x) - kL| = |k|\,|f(x) - L| < |k|\,(\epsilon/|k|) = \epsilon.$$