## Optional Lecture 2: Area between curves

## Objectives:

(O2.1) Find the area of a bounded region between the graphs of two functions.

## Area between curves

Ever since its introduction, we have used the definite integral to compute area. In most cases, the region under consideration was bounded by the graph of a positive-valued function, the $x$-axis, and two vertical lines. In this lecture we consider regions between graphs such as that illustrated below.


If we interpret each of the integrals

$$
\int_{a}^{b} f(x) d x \quad \text { and } \quad \int_{a}^{b} g(x) d x
$$

as giving the area under each graph, respectively, then it is easy to see that the area of the shaded region is given by

$$
\text { Area }=\int_{a}^{b} f(x) d x-\int_{a}^{b} g(x) d x=\int_{a}^{b}[f(x)-g(x)] d x
$$

A nice thing about this new idea is that we no longer have to worry about our functions having positive values. If $f(x) \geq g(x)$, then we must have $f(x)-g(x) \geq 0$, regardless of the functions' individual values.

## Theorem 1 - Area between curves

If $f$ and $g$ are continuous functions on $[a, b]$ with $f(x) \geq g(x)$, then the area between the graphs of $f$ and $g$ on $[a, b]$ is given by

$$
\text { Area }=\int_{a}^{b}[f(x)-g(x)] d x
$$

Example 1 Find the area of the bounded region between the graphs of $y=x^{2}$ and $y=x^{3}$.
The graphs of $y=x^{2}$ and $y=x^{3}$ intersect when

$$
x^{2}-x^{3}=x^{2}(1-x)=0 \quad \text { or } \quad x=0,1
$$

The bounded region extends from $x=0$ to $x=1$, and on this interval $x^{2} \geq x^{3}$.


The area is given by

$$
\text { Area }=\int_{0}^{1}\left(x^{2}-x^{3}\right) d x=\left[\frac{1}{3} x^{3}-\frac{1}{4} x^{4}\right]_{0}^{1}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}
$$

Example 2 Find the area of the region between the graphs of $f(x)=\sin x$ and $g(x)=\cos x$ over the interval $[0,1]$.

The region is shown below.

|  | y |
| :---: | :---: |
|  | $y=\cos (x)$ |
|  |  |
|  | X |

The graphs of $f$ and $g$ intersect at $x=\pi / 4 \approx 0.785$, and at this point the top graph and bottom graph switch. We must take this switch into account when computing the area:

$$
\text { Area }=\int_{0}^{\pi / 4}(\cos x-\sin x) d x+\int_{\pi / 4}^{1}(\sin x-\cos x) d x=2 \sqrt{2}-1-\sin (1)-\cos (1) \approx 0.45
$$

where the details have been omitted.
Although we could not have manually computed the area using a single integral, we could have
described the area with one integral:

$$
\int_{0}^{1}|\sin x-\cos x| d x
$$

Wolfram Alpha could be used to evaluate this integral.

Example 3 Find the area of the region between the graphs of $r(x)=x^{3}-2 x$ and $s(x)=x^{2}$.


The details are omitted, but we should find

$$
\text { Area }=\int_{-1}^{0}[r(x)-s(x)] d x+\int_{0}^{2}[s(x)-r(x)] d x=\frac{5}{12}+\frac{8}{3}=\frac{37}{12}
$$

Example 4 Find the positive number $k$ such that the area of the region between the graphs of $y=k x$ and $y=x^{2}$ is 1 .

Let $k$ represent some unknown positive number. The graphs of $y=k x$ and $y=x^{2}$ intersect when

$$
k x-x^{2}=x(k-x)=0 \quad \text { or } \quad x=0, k
$$

On the interval $[0, k], k x \geq x^{2}$. Therefore the area of the region between the graphs is given by

$$
\text { Area }=\int_{0}^{k}\left(k x-x^{2}\right) d x=\left[\frac{k}{2} x^{2}-\frac{1}{3} x^{3}\right]_{0}^{k}=\frac{k^{3}}{2}-\frac{k^{3}}{3}=\frac{k^{3}}{6}
$$

It follows that

$$
\text { Area }=1 \quad \Longrightarrow \quad k=\sqrt[3]{6}
$$

