Section 3.7 - Improper Integrals

So far we have only seen integrals of the form

$$\int_a^b f(x)\,dx,$$

where f is continuous on the closed and bounded interval $[a, b]$. What if f is not continuous or the interval is unbounded?

Example 1 (for motivation)

Think about $\displaystyle\int_0^t e^{-x}\,dx$ for any $t > 0$.

The integrand, $f(x) = e^{-x}$ is continuous on $[0, t]$ for any positive number t. Therefore, the integral exists for any number t, and

$$\int_0^t e^{-x}\,dx = -e^{-x}\Big|_0^t = -e^{-t} + 1.$$

Now notice that

$$\lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}(-e^{-t} + 1) = 1.$$

This motivates the following notation and result:

$$\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = 1. \quad \diamondsuit$$

Definition

An <u>improper integral</u> is an integral (1) on an unbounded interval or (2) whose integrand has a finite number of infinite discontinuities in the integration interval. Improper integrals may also be of both types.

Definition for Case 1 -- Improper integrals on unbounded intervals

a.) If f is continuous on $[a, \infty)$, then $\displaystyle\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx.$

b.) If f is continuous on $(-\infty, b]$, then $\displaystyle\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx.$

c.) If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx,$$

for any real number c, where the integrals on the right are limits from b.) and a.), respectively.

If any one of the limits on the right does not exist, we say that the improper integral <u>diverges</u>.

Example 2

Rewrite as a limit and evaluate: $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

This improper integral is an example of case 1a.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left(\ln|x| \Big|_1^t \right) = \lim_{t\to\infty} \ln t = \infty.$$

The integral diverges. $\diamondsuit$

Example 3

Rewrite as a limit and evaluate: $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+1}\,dx$

This improper integral is an example of case 1b.

$$\int_{-\infty}^0 \frac{1}{x^2+1}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{x^2+1}\,dx = \lim_{t\to-\infty} (\tan^{-1} x)\Big|_t^0 = \lim_{t\to-\infty} (0 - \tan^{-1} t) = \frac{\pi}{2}. \quad \diamondsuit$$

Example 4

Evaluate the improper integral: $\displaystyle\int_{-\infty}^\infty \sin x\,dx$

This improper integral is an example of case 1c.

$$\int_{-\infty}^\infty \sin x\,dx = \int_{-\infty}^0 \sin x\,dx + \int_0^\infty \sin x\,dx$$

The zero for the bound could have actually been any real number. Let's start with the right-most integral.

$$\int_0^\infty \sin x\,dx = \lim_{t\to\infty} \int_0^t \sin x\,dx = \lim_{t\to\infty} (-\cos x)\Big|_0^t = \lim_{t\to\infty} (-\cos t + 1)$$

Since the cosine function oscillates, the limit does not exist. Therefore the integral diverges. Since the right-most integral diverges, the original improper integral diverges. There is no need to even check the integral with the $-\infty/0$-bounds. $\diamondsuit$

Definition for Case 2 -- Integrands with infinite discontinuities

a.) If f is continuous on $[a, b)$ and has a discontinuity at b, then $\displaystyle\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx.$

b.) If f is continuous on $(a, b]$ and has a discontinuity at a, then $\displaystyle\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx.$

c.) If f is continuous on $[a, c) \cup (c, b]$ and has a discontinuity at c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

for any real number c, where the integrals on the right are limits from b.) and a.), respectively.

If any one of the limits on the right does not exist, we say that the improper integral diverges.

Example 5

Rewrite as a limit and evaluate: $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$

The integrand has an infinite discontinuity at the lower bound, $x = 0$. This is an example of case 2b.

$$\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^+} \int_t^1 x^{-1/3}\, dx = \lim_{t\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_t^1 = \lim_{t\to 0^+} \left(\frac{3}{2} - \frac{3}{2}t^{2/3}\right) = \frac{3}{2} \qquad \diamondsuit$$

Example 6

Rewrite as a limit and evaluate: $\displaystyle\int_{-1}^0 \frac{1}{x^3}\, dx$

The integrand has an infinite discontinuity at the upper bound, $x = 0$. This is an example of case 2a.

$$\int_{-1}^0 \frac{1}{x^3} = \lim_{t\to 0^-} \int_{-1}^t x^{-3}\, dx = \lim_{t\to 0^-} \left[-\frac{1}{2}x^{-2}\right]_{-1}^t = \lim_{t\to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2}\right) = -\infty$$

The integral diverges. $\diamondsuit$

Example 7

Evaluate the improper integral: $\displaystyle\int_{-1}^2 \frac{1}{x^3}\, dx$

The integrand has an infinite discontinuity at $x = 0$, inside the integration interval. This is an example of case 2c.

$$\int_{-1}^2 \frac{1}{x^3} = \int_{-1}^0 x^{-3}\, dx + \int_0^2 x^{-3}\, dx$$

Now, the first integral on the right is divergent integral from example 6. We need not check any further---the original integral must diverge.

Be very careful with integrals of type 2c! If we would not have noticed the discontinuity at $x = 0$, **we might have incorrectly evaluated** as follows:

$$\int_{-1}^2 \frac{1}{x^3} = \left[-\frac{1}{2}x^{-2}\right]_{-1}^2 = \left(-\frac{1}{8} + \frac{1}{2}\right) = \frac{3}{8}.$$

This is terribly wrong! $\diamondsuit$