Section 7.1 - Parametric Equations

Example 1 (for motivation)

The graph of the equation $x^2 + y^2 = 1$ is the unit circle centered at the origin. We know from the definitions of the trigonometric functions that the x- and y-coordinates of the points on the unit circle are described by cosines and sines, respectively. In fact, for any point (x, y) on the unit circle, there is an angle θ such that

$$x = \cos\theta, \quad y = \sin\theta, \qquad 0 \le \theta < 2\pi.$$

There is an interesting and very important idea that you may have forgotten since your trig class: when the angle θ is measured in radians, it represents an arc length along the unit circle, counterclockwise from the point $(1, 0)$ to (x, y).

Notice that the equations $x = \cos\theta$ and $y = \sin\theta$ describe points on the unit circle just as well as (if not better than) the equation $x^2 + y^2 = 1$. The trigonometric functions describe the points in terms of the *parameter* θ.

Definition 1

If x and y are continuous functions of t on an interval I, then

$$x = x(t) \qquad \text{and} \qquad y = y(t)$$

are called <u>parametric equations</u> and t is called the <u>parameter</u>. The set of all (x, y)- points obtained as t varies over the interval I is called the <u>graph of the parametric equations</u>.

Example 2

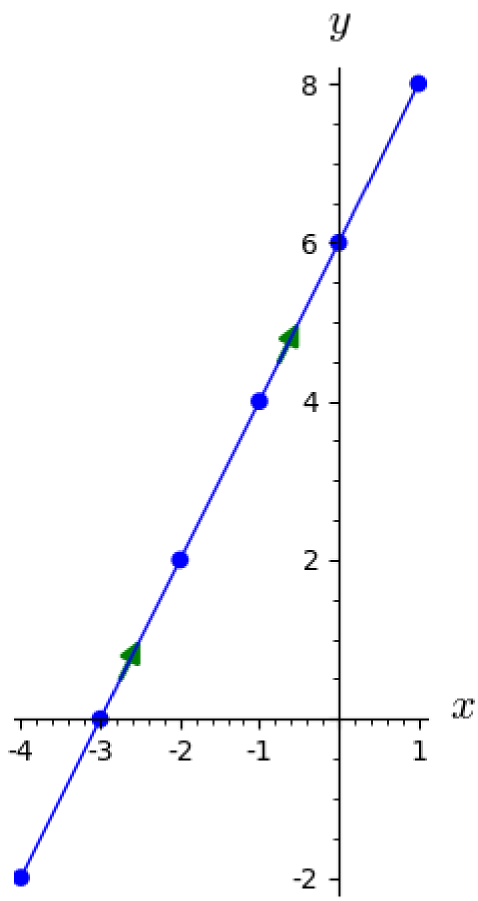
Sketch the graph of the parametric equations:

$$x = t - 1, \quad y = 2t + 4, \qquad -3 \le t \le 2.$$

Because we are new to parametric equations, we will probably have to get our ideas about the graph from plotting points.

t	$x(t)$	$y(t)$
-3	-4	-2
-2	-3	0
-1	-2	2
0	-1	4
1	0	6
2	1	8

Plot the points and draw a smooth curve...



An important feature of a parametric curve is that it has an *orientation*, that is, a direction associated with increasing t. The orientation is indicated above by the arrows.

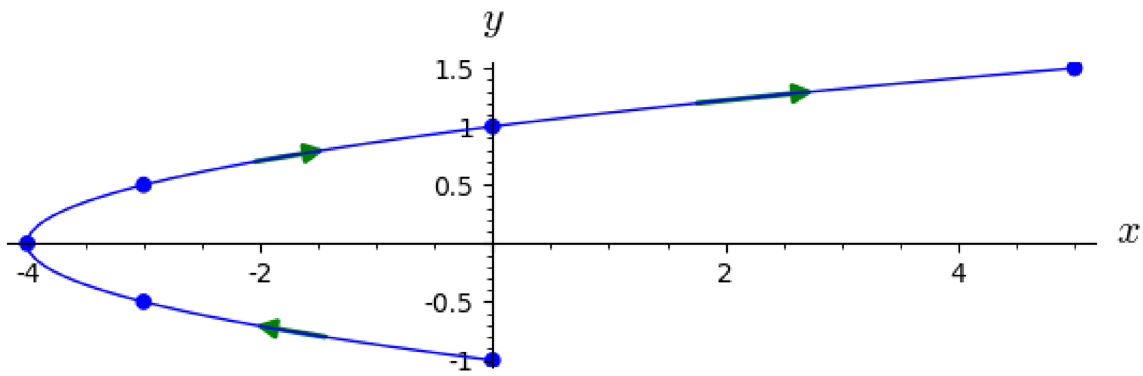
Since x and y both change linearly with t, it is probably not surprising that the graph is a line segment. ♢

Example 3

Sketch the graph of the parametric equations:

$$x = t^2 - 4, \quad y = \frac{t}{2}, \qquad -2 \leq t \leq 3.$$

t	$x(t)$	$y(t)$
-2	0	-1
-1	-3	-1/2
0	-4	0
1	-3	1/2
2	0	1
3	5	3/2



Based on the equations, it is probably not surprising that the graph is a parabola. ♢

Comments

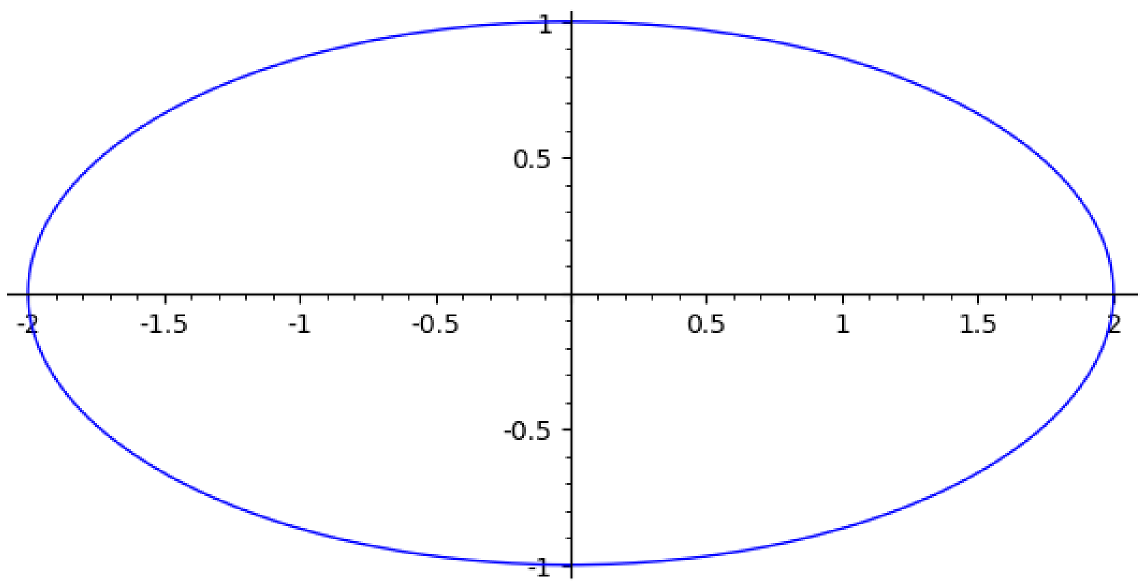
1. You can graph parametric equations on your TI graphing calculator by switching to parametric mode.
2. Computer algebra systems can also plot in parametric mode. The SageMath syntax for graphing parametric equations is `var("t"); parametric_plot((x(t),y(t)),(t,a,b))`. For example, to sketch the curve in example 3, you would use `"var("t"); parametric_plot((t^2-4,t/2),(t,-2,3))`

Example 4

Use a graphing utility to sketch the graph of the parametric equations:

$$x = -2\sin t, \quad y = \cos t, \qquad 0 \leq t < 2\pi.$$

Using the SageMath syntax `var("t"); parametric_plot((-2*sin(t),cos(t)),(t,0,2*pi))`, the following graph was obtained.



The parametric equations describe an ellipse. You should spend a few moments to determine the orientation of the curve. ♢

Eliminating the parameter

In example 1, we saw that the equation $x^2 + y^2 = 1$ and the parametric equations $x = \cos t$, $y = \sin t$ have the same graph. (We actually used the parameter θ in example 1, but the symbol we use for the parameter is irrelevant.) It is often more convenient to have a curve described in terms of a parameter, rather than by a two-variable equation. However, it is sometimes useful to convert from a set of parametric equations to a two-variable equation. This is called *eliminating the parameter*. There are no magic tricks that always work for eliminating parameters, but here are two common strategies:

1. Solve one equation for the parameter and substitute into the other equation.
2. Use the Pythagorean identities for the trig functions.

Example 5

Eliminate the parameter: $\quad x = t + 3, \quad y = 2t, \quad -5 \le t \le 5$

From the first equation, it follows that $t = x - 3$. Substituting into the second equation gives $y = 2(x - 3) = 2x - 6$. In this example, it is also easy to adjust the domain. The t-interval $-5 \le t \le 5$ coincides precisely with the x-interval $-2 \le x \le 8$. So the graph of the parametric equations is identical to the graph of the

$$y = 2x - 6, \quad -2 \le x \le 8. \qquad \diamondsuit$$

Example 6

Eliminate the parameter: $\quad x = 3\cos\theta, \quad y = 4\sin\theta, \quad 0 \le \theta < 2\pi$

Rewrite

$$\cos\theta = \frac{x}{3} \qquad \text{and} \qquad \sin\theta = \frac{y}{4},$$

and then use a Pythagorean identity:

$$\cos^2\theta + \sin^2\theta = \frac{x^2}{9} + \frac{y^2}{16} = 1.$$

The parametric equations describe the ellipse centered at $(0, 0)$ with vertices $(\pm 3, 0)$ and $(0, \pm 4)$. $\diamondsuit$

Parameterizing a curve

Just as we can eliminate a parameter in parametric equations, we can introduce a parameter for a two-variable equation. There are lots of ways to do so. In practice, the parameter may have a physical meaning or significance. For example, the parameter θ in example 1 represents arc length. On the other hand, sometimes it is convenient to obtain a trivial parameterization by simply letting the parameter represent one of the coordinates.

Example 7

Find a set of parametric equations for the line segment described by $y = 3x + 5$, $0 \le x \le 4$.

Let's explore two very simple options.

1. Introduce the parameter t by letting $x = t$. Then we have the following parametric equations:

   $$x = t, \quad y = 3t + 5, \qquad 0 \le t \le 4.$$

   In this case, the parameter t represents the x-coordinate.

2. Introduce the parameter t by letting $y = t$.

$$x = \frac{t-5}{3}, \quad y = t, \qquad 5 \leq t \leq 17.$$

In this case, the parameter represents the y-coordinate. Do you understand how x was obtained? And do you see why the t-interval is $[5, 17]$? $\diamondsuit$

Example 8

Parameterize the graph $y = x^2$ in terms of the slope of the curve at any point.

At the point (x, y), the slope of the curve is given by the derivative $\frac{dy}{dx} = 2x$. Let m be the derivative so that $m = 2x$. It follows that $x = m/2$ and $y = m^2/4$. So the parametric equations, in terms of slope m, are

$$x = \frac{m}{2} \quad \text{and} \quad y = \frac{m^2}{4}.$$

Now imagine we wanted to know the point on the graph of $y = x^2$ at which the derivative is 6. The parametric equations are designed for this type of problem. We simply plug in $m = 6$ to get the point $(3, 9)$. $\diamondsuit$