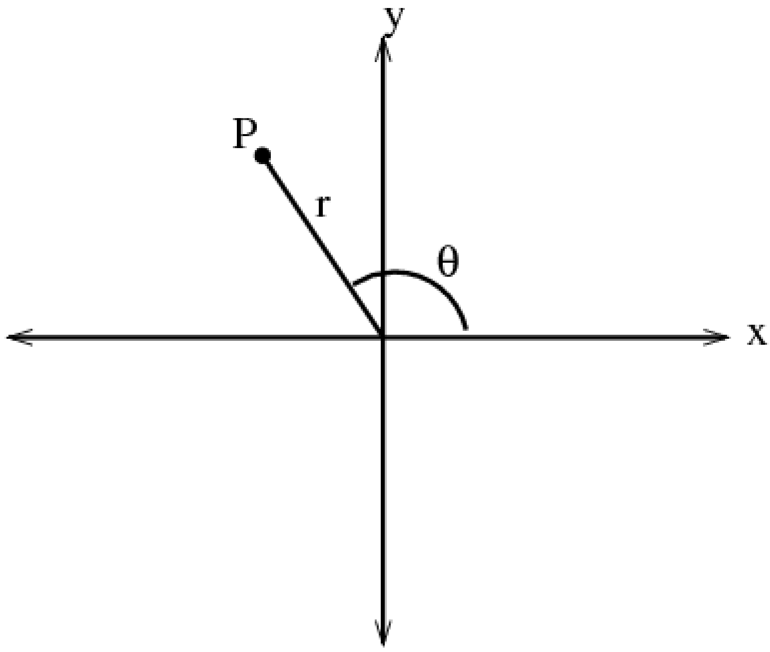
Section 7.3 - Polar Coordinates

The rectangular (or Cartesian) coordinate system provides a means for identifying points in the plane with ordered pairs of numbers. In other words, it provides a technique for *mapping* physical points to ordered pairs. The rectangular coordinate system is called "rectangular" because of the grid pattern invoked by plotting points---we count over and up from the origin. The *polar coordinate system* provides an alternative to rectangular coordinates. In certain cases, polar coordinates are far more natural and convenient than rectangular coordinates.

In polar coordinates, a point P, in the plane, is identified by an order pair of numbers (r, θ), where r is the signed distance from the origin and θ is the angle that the ray from the origin to P makes with the positive x-axis (the *polar axis*).



The polar coordinates for a point in the plane are not unique. For example, the point whose rectangular coordinates are $(x, y) = (1, 0)$ can be represented in polar coordinates in infinitely many ways. Here are a few of them:

$$(1,0), \qquad (1,2\pi), \qquad (-1,\pi), \qquad (-1,27\pi).$$

In practice, we will often choose r and θ such that $r \geq 0$ and $0 \leq \theta < 2\pi$, but this will rarely be required.

Theorem 1

Suppose P is a point in the plane with rectangular coordinates (x, y) and polar coordinates (r, θ). Then

$$\begin{aligned} x &= r\cos\theta, & y &= r\sin\theta \\ r^2 &= x^2 + y^2, & \tan\theta &= \frac{y}{x}. \end{aligned}$$

Example 1

Convert the point $(x, y) = (1, 1)$ to polar coordinates.

Notice that the point is in the 1st quadrant, so we must choose r and θ appropriately.

By the conversion formulas, we have

$$x^2 + y^2 = 1^2 + 1^2 = 2 \quad \Longrightarrow \quad r^2 = 2$$

and

$$\tan\theta = 1.$$

In the 1st quadrant, we can choose

$$r = \sqrt{2}, \qquad \theta = \tan^{-1}(1) = \frac{\pi}{4}.$$

The point can be represented by the polar coordinates $(r, \theta) = (\sqrt{2}, \pi/4)$. $\diamondsuit$

Example 2

Convert the point $(x, y) = (-4, 7)$ to polar coordinates.

This point is in the 2nd quadrant. By the conversion formulas, we have

$$(-4)^2 + 7^2 = 65 \quad \Longrightarrow \quad r^2 = 65$$

and

$$\tan\theta = -\frac{7}{4}.$$

Let's choose $r = \sqrt{65}$. Since $\tan^{-1}(-7/4)$ is in the 4th quadrant, we will have to add π to get an angle in the 2nd quadrant:

$$\theta = \pi + \tan^{-1}(-7/4) = 2.089942.$$

The point can be represented by the polar coordinates

$$(r, \theta) = (\sqrt{65}, \pi + \tan^{-1}(-7/4)) \approx (8.06, 2.09). \qquad \diamondsuit$$

Example 3

Convert the point $(r, \theta) = (6, -5\pi/6)$ to rectangular coordinates.

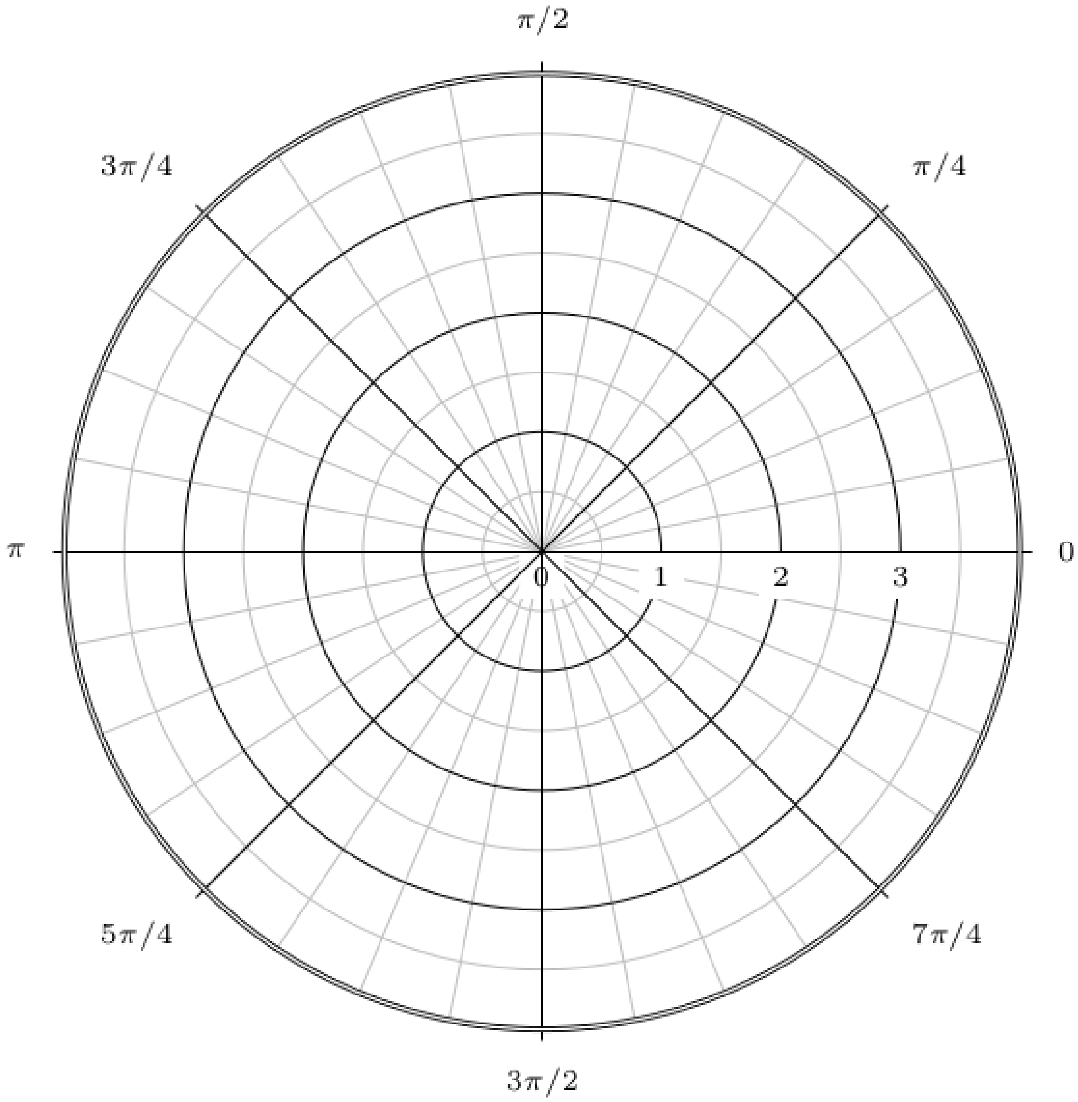
Converting from polar to rectangular is easy.

$$x = 6\cos(-5\pi/6) = 6\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$$
$$y = 6\sin(-5\pi/6) = 6\left(-\frac{1}{2}\right) = -3$$

In rectangular coordinates, the point is $(-3\sqrt{3}, -3)$. $\diamondsuit$

Graphing in polar coordinates

When graphing in the rectangular coordinate system, we think of the system as being constructed of a rectangular grid. In order to plot points, we count over and up along grid squares. In polar coordinates, we do not count over and up, rather we count out from the origin, and we sweep out an angle from the polar axis. A rectangular grid is not useful for describing this new process, so we use a polar grid.



On a polar grid, concentric circles indicate specific distances from the origin, and radial lines indicate specific angles from the polar axis. To plot the point $(r, \theta) = (3, 5\pi/4)$, we locate the intersection of the circle of radius 3 with the radial line that makes the angle of $5\pi/4$ with the polar axis. Can you find that location?

Equations in polar coordinates have graphs in the polar coordinate system. Until you become familiar with polar equations and their graphs, you should sketch polar curves by plotting points and using technology.

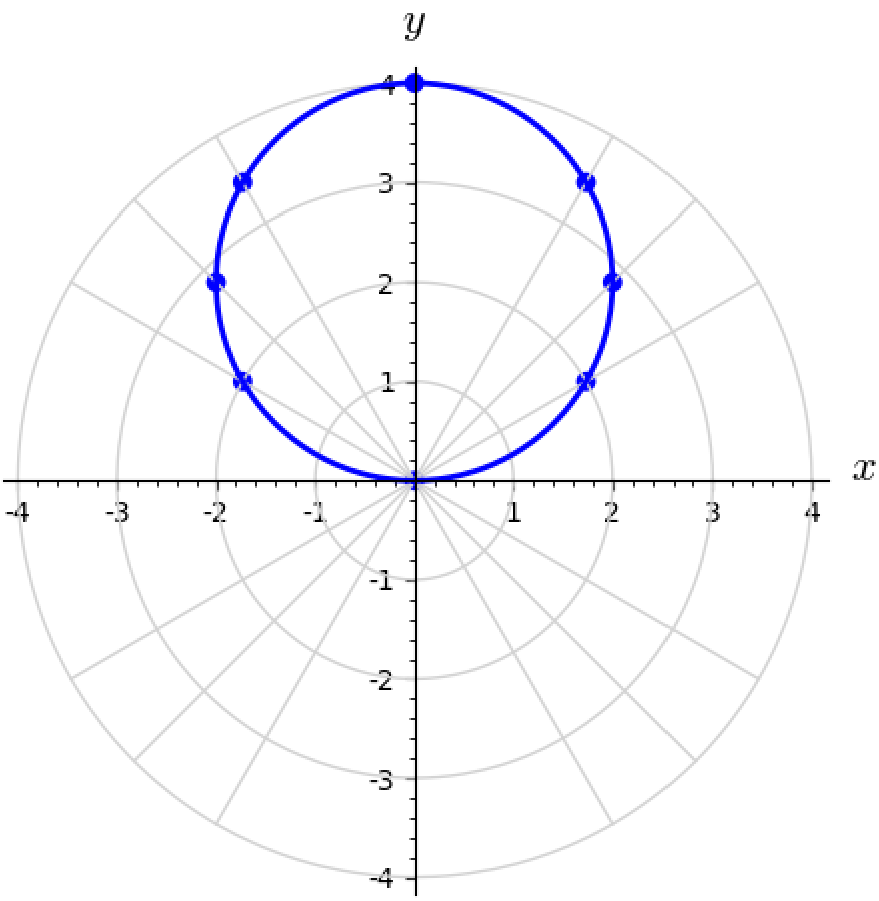
Example 4

Sketch the graph of the polar curve $r = 4\sin\theta$.

Let's make a table of values so that we can plot points. Since r is a function of θ, we choose some values for θ and compute the corresponding r values.

θ	r
0	0
$\pi/6$	2
$\pi/4$	$2\sqrt{2} \approx 2.8$
$\pi/3$	$2\sqrt{3} \approx 3.4$
$\pi/2$	4
$2\pi/3$	$2\sqrt{3} \approx 3.4$
$3\pi/4$	$2\sqrt{2} \approx 2.8$
$5\pi/6$	2
π	0

We could continue the table by further incrementing θ, but we would not find any new points. Now let's plot the points, look for a pattern, and draw a smooth curve. The graph is shown below.



Comments

1. You can graph polar equations on your TI graphing calculator by switching to polar mode.
2. Computer algebra systems can also plot in polar mode. The SageMath syntax for graphing polar equations is `var("theta"); polar_plot(r(theta),(theta,a,b))`. For example, to sketch the curve in example 4, you would use `var("theta"); polar_plot(4*sin(theta),(theta,0,pi))`.

3. With experience, you will become familiar with certain, common polar curves. You can find a table of common curves in our textbook . There is also a nice reference sheet available at https://faculty.math.illinois.edu/~ecaulfi2/231su17/polar-curves.pdf.
4. It is sometimes useful to convert equations from polar coordinates to rectangular coordinates or vice versa. To do so, use the conversion formulas in theorem 1, along with some algebra and trig.

Example 5

Convert the equation $r = 4\sin\theta$ to an equation in rectangular coordinates.

The easy way to convert this equation is to start by multiplying both sides by r:

$$r^2 = 4r\sin\theta.$$

Now use the conversion formulas and complete the square.

$$\begin{aligned} x^2 + y^2 &= 4y \\ x^2 + y^2 - 4y &= 0 \\ x^2 + y^2 - 4y + 4 &= 4 \\ x^2 + (y-2)^2 &= 2^2 \end{aligned}$$

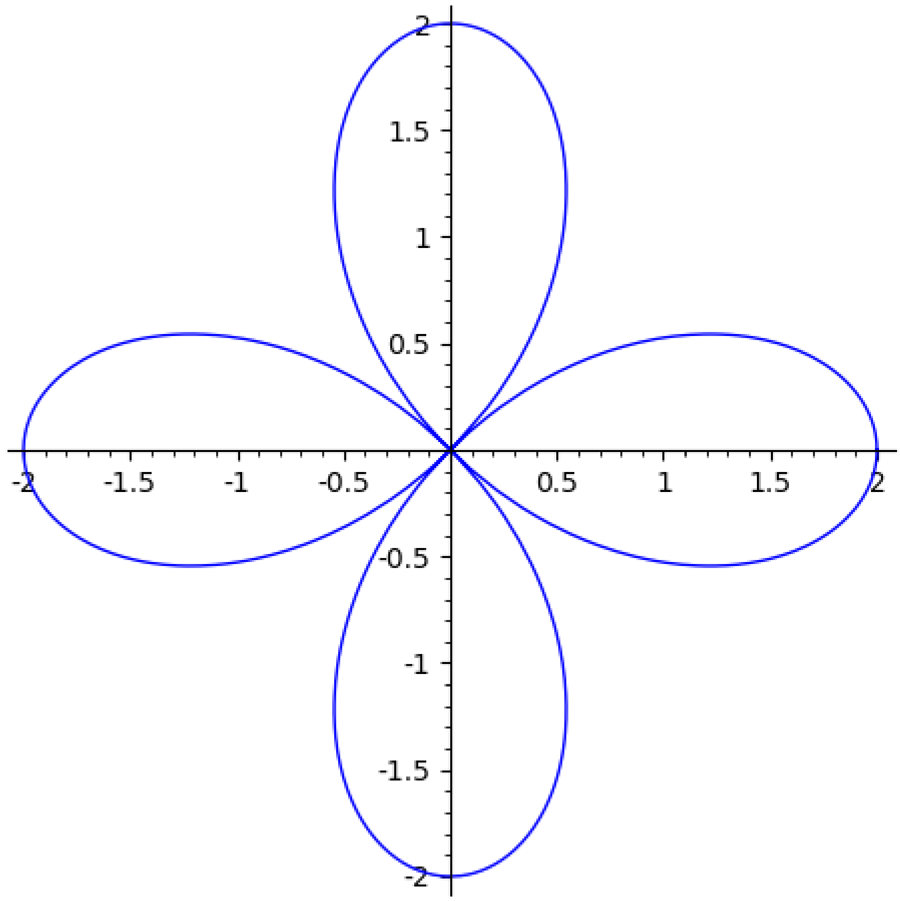
This familiar rectangular equation describes the circle of radius 2 centered at $(0, 2)$. That circle was exactly what we graphed in example 4. $\diamondsuit$

Example 6

Sketch the graph of the polar curve $r = 2\cos(2\theta)$.

This is a very common polar curve known as a *rose curve*. We'll use technology to sketch the graph. We get the graph below by using the following SageMath code:

```
var("theta")
polar_plot(2*cos(2*theta),(theta,0,2*pi))
```

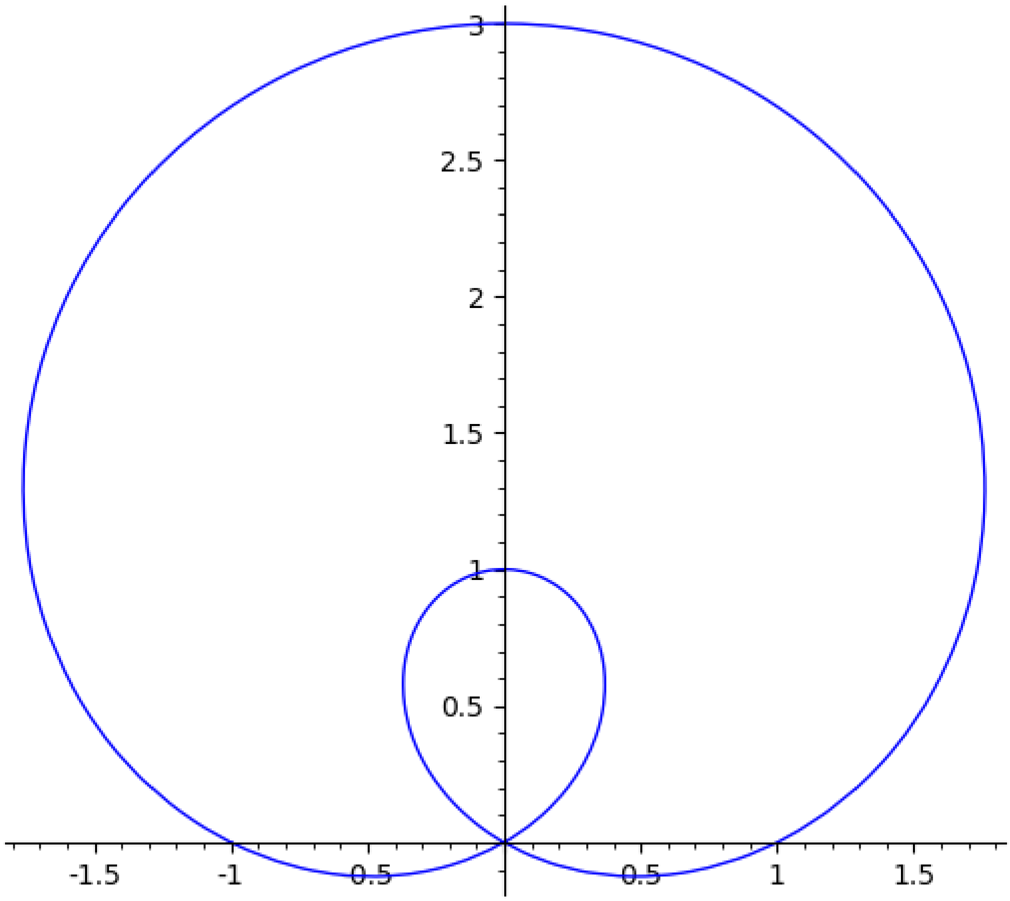


Example 7

Sketch the graph of the polar curve $r = -1 + 2\sin\theta$.

This is also a very common polar curve. We get the graph below by using the following SageMath code:

```
var("theta")
polar_plot(-1+2*sin(theta),(theta,0,2*pi))
```



Symmetry tests for polar equations

When graphing in polar coordinates, it can be very useful to take advantage of symmetry. Here are some symmetry tests that we can apply:

1. In a polar equation, replace every θ by $-\theta$ and then simplify. If the equation is unchanged, then the graph is symmetric about the polar axis (i.e., the positive x-axis).
2. In a polar equation, replace every θ by $\pi - \theta$ and then simplify. If the equation is unchanged, then the graph is symmetric about the line $\theta = \pi/2$ (i.e., the positive y-axis).
3. In a polar equation, replace every r by $-r$ and then simplify. If the equation is unchanged, then the graph is symmetric about the origin. (Equivalently, if θ is replaced by $\pi + \theta$, and the equation is unchanged, the graph is symmetric abut the origin.)