Math 157 - Final Exam

December 11, 2013

Name Key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Let $f(x) = (x^2 - x + 1)^2$. Find an equation of the line tangent to the graph of f at the point where $x = 2$.

$f'(x) = 2(x^2 - x + 1)(2x - 1)$

$m = f'(2) = 2(3)(3) = 18$

$f(2) = (3)^2 = 9$

TAN LINE IS

$y - 9 = 18(x - 2)$

OR

$y = 18x - 27$

2. (8 points) Find k so that g is continuous at $x = 2$.

$$g(x) = \begin{cases} 8x + k, & x < 2 \\ x^2 + \sqrt{x+2}, & x \geq 2 \end{cases}$$

$\lim_{x \to 2^-} g(x) = g(2)$

$8(2) + k = 4 + \sqrt{4}$

$16 + k = 6$

$\boxed{k = -10}$

3. (12 points) Suppose y is implicitly defined as a function of x by the equation

$$6x^2 = y^2 + xy.$$

Find dy/dx at $(1, 2)$.

$$\frac{d}{dx}(6x^2) = \frac{d}{dx}(y^2 + xy)$$

$$12x = 2y\frac{dy}{dx} + y + x\frac{dy}{dx}$$

$$12x - y = (2y + x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{12x - y}{2y + x}$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{10}{5} = \boxed{2}$$

4. (12 points) Find each limit analytically. Use ∞, $-\infty$, or DNE if appropriate.

(a) $\lim_{x \to 8^-} (x^2 - 9x + 7)$

$$= 8^2 - 9(8) + 7 = 64 - 72 + 7 = \boxed{-1}$$

$0/0$

(b) $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)(x+3)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x+3}{x+2} = \boxed{\frac{5}{4}}$

$-4/0$ INFINITE LIMIT!

(c) $\lim_{x \to 4^+} \frac{x - 8}{x - 4} = \boxed{-\infty}$

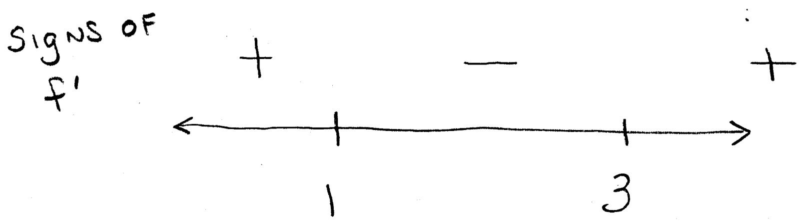
FOR $x > 4$ BUT CLOSE TO 4: $\frac{x-8}{x-4} = \frac{\text{NEG}}{\text{POS}} = \text{NEG}$

5. (20 points) Consider the function $f(x) = 2x^3 - 12x^2 + 18x$.

(a) Find all critical numbers of f.

$$f'(x) = 6x^2 - 24x + 18$$
$$= 6(x^2 - 4x + 3)$$
$$= 6(x-3)(x-1) \stackrel{\cdot}{=} 0$$
$$\Rightarrow \boxed{x = 3,\ x = 1}$$

(b) Find open intervals on which f is increasing/decreasing.



f is increasing on $(-\infty, 1) \cup (3, \infty)$

f is decreasing on $(1, 3)$

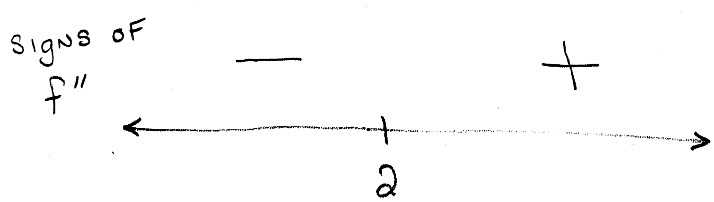
(c) Find the relative extreme values of f.

$f(1) = 8$ is a rel max

$f(3) = 0$ is a rel min

(d) Find open intervals on which the graph of f is concave up/down.

$$f''(x) = 12x - 24 = 0$$
$$\Rightarrow x = 2$$



Graph is CD on $(-\infty, 2)$

Graph is CU on $(2, \infty)$

6. (6 points) The profit P (in dollars) for producing x units of a product is given by $P = -2x^2 + 72x - 145$. Find the production level at which the marginal profit is zero.

$$P'(x) = -4x + 72$$

$$-4x + 72 = 0$$

$$\Rightarrow \quad x = \frac{72}{4} = 18$$

$$\boxed{x = 18}$$

7. (10 points) Let $f(x) = x^2 e^{-2x}$. Find $f''(x)$.

$$f'(x) = 2xe^{-2x} - 2x^2e^{-2x}$$

$$f''(x) = 2e^{-2x} - 4xe^{-2x} - 4xe^{-2x} + 4x^2e^{-2x}$$

$$= \boxed{(4x^2 - 8x + 2)e^{-2x}}$$

8. (8 points) The revenue for a company selling x units of a product is

$$R = 1200x - 0.15x^2.$$

Use differentials to estimate the change in revenue as the sales increase from 2000 units to 2025 units.

$$dR = (1200 - 0.3x)\,dx$$

$$\Delta R \approx (1200 - 0.3x)\,\Delta x$$

$$\Delta R \approx (1200 - 0.3(2000))(25)$$

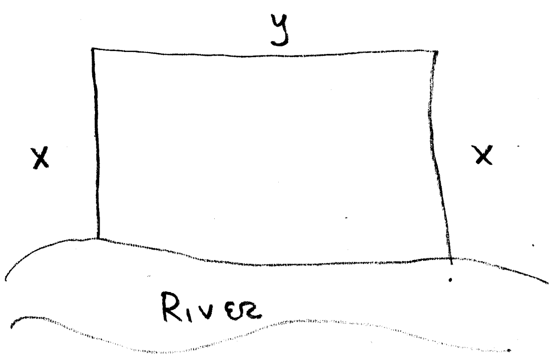
$$= \boxed{15000}$$

FOR COMPARISON, THE ACTUAL CHANGE IS

$$R(2025) - R(2000)$$

$$= 14906.25$$

9. (12 points) A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must have an area of 88,200 square meters. No fencing is needed along the river. What dimensions will require the least amount of fencing?



Minimize $P = 2x + y$

s.t. $xy = 88200$

$y = \dfrac{88200}{x}$

$P(x) = 2x + \dfrac{88200}{x}, \quad x > 0$

$P'(x) = 2 - \dfrac{88200}{x^2} = 0$

$\Rightarrow x = 210$

$P''(x) = \dfrac{176400}{x^3}$

$P''(210) > 0 \Rightarrow x = 210$ gives a min.

Dimensions are $x = 210$ m. and $y = \dfrac{88200}{210}$ 420m

10. (8 points) Determine the derivative of each function. It may be helpful to use exponent laws, logarithm laws, or change-of-base formulas to simplify the functions before differentiation.

(a) $g(t) = \ln[t^3(t+5)^2]$

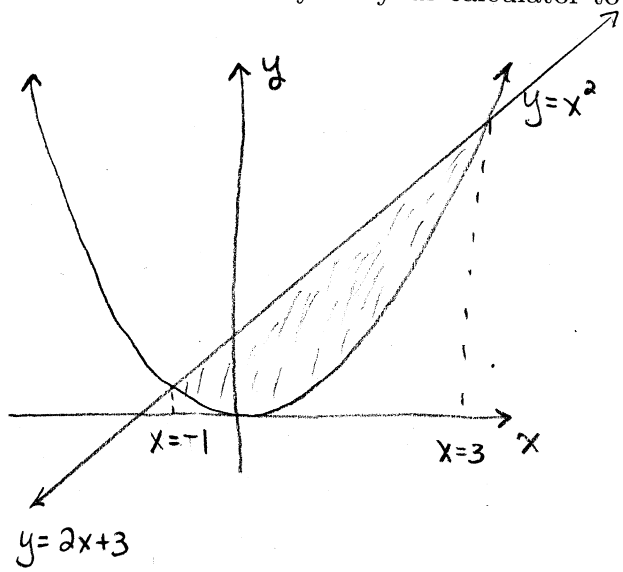
$g(t) = 3\ln t + 2\ln(t+5)$

$g'(t) = \dfrac{3}{t} + \dfrac{2}{t+5}$

(b) $f(x) = \log_2 x = \dfrac{\ln x}{\ln 2}$

$f'(x) = \dfrac{1}{\ln 2} \dfrac{1}{x}$

11. (12 points) Find the area of the region bounded by the graphs of $y = x^2$ and $y = 2x+3$. You may use your calculator to evaluate your definite integral.



$$x^2 = 2x+3$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3 \quad x = -1$$

$$\int_{-1}^{3} \left[(2x+3) - x^2\right] dx$$

$$= \frac{32}{3} = 10.\overline{6}$$

12. (8 points) Evaluate the definite integral: $\displaystyle\int_{-1}^{1} 3x^2\,(x^3+1)^4\,dx$.

$$u = x^3 + 1 \qquad x = -1 \Rightarrow u = 0$$

$$du = 3x^2\,dx \qquad x = 1 \Rightarrow u = 2$$

$$\int_0^2 u^4\,du = \frac{1}{5}u^5\Big|_0^2 = \boxed{\frac{32}{5}}$$

13. (10 points) Evaluate the indefinite integral: $\int \left(6x^2 - \frac{6}{x} + 2e^x\right)\, dx$

$$= \boxed{2x^3 - 6\ln|x| + 2e^x + C}$$

14. (8 points) The half-life of radioactive radium (^{226}Ra) is 1599 years. How long will it take a sample of 150 g to decay to 8 g?

$P(t) = P_0 e^{kt}$

$\frac{1}{2}P_0 = P_0 e^{k(1599)} \Rightarrow k = \frac{\ln 1/2}{1599}$

$\boxed{t \approx 6761.88 \text{ years}}$

$8 = 150e^{kt}$

$\Rightarrow \frac{8}{150} = e^{kt} \Rightarrow \frac{\ln \frac{8}{150}}{k} = t$

15. (6 points) Use integration by parts to evaluate the indefinite integral: $\int (x^2 \ln x)\, dx$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g'(x) = x^2 \qquad g(x) = \frac{1}{3}x^3$

$$\int x^2 \ln x\, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2\, dx$$

$$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$