Math 157 - Final Exam

December 9, 2015

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, all limits, derivatives, and integrals should be computed by hand with all work shown.

1. (10 points) Let $f(x) = \ln(x^2 - 4x + 2)$. Compute $f'(x)$. Then use it to find an equation of the line tangent to the graph of f at the point where $x = 0$.

Slope: $f'(x) = \frac{1}{x^2-4x+2}(2x-4)$

$m = f'(0) = \frac{-4}{2} = -2$

Point: $y = f(0) = \ln 2$

$(0, \ln 2)$

Line:

$y - \ln 2 = -2(x-0)$

$y = -2x + \ln 2$

2. (8 points) Use algebra to evaluate the limit.

$$\lim_{x\to 3} \frac{x^2+2x-15}{2x-6} = \lim_{x\to 3} \frac{(x+5)\cancel{(x-3)}}{2\cancel{(x-3)}}$$

$$= \frac{8}{2} = \boxed{4}$$

3. (6 points) Find the critical point(s) of $h(x) = (8x+6)^{2/3}$.

$h'(x) = \frac{2}{3}(8x+6)^{-1/3}(8)$

$= \frac{16/3}{\sqrt[3]{8x+6}}$

$h'(x)$ DNE when $8x+6=0$

$x = \frac{-6}{8} = \frac{-3}{4}$

Only crit pt is at $x = -3/4$

4. (20 points) Consider the function $f(x) = -2x^3 - 3x^2 + 36x - 3$.

(a) Find all critical numbers of f.

$$f'(x) = -6x^2 - 6x + 36$$
$$= -6(x^2 + x - 6)$$
$$= -6(x+3)(x-2).$$

$f'(x) = 0 \Rightarrow$ $\boxed{x = -3,\ x = 2}$

(b) Find open intervals on which f is increasing/decreasing.

SIGNS OF f': $-$ | $+$ | $-$ at -3, 2

f: DEC | INC | DEC

f IS INCREASING ON $(-3, 2)$

AND DECREASING ON $(-\infty, -3) \cup (2, \infty)$

(c) Find all local (relative) extreme values of f.

$f(-3) = -84$ IS A REL MIN

$f(2) = 41$ IS A REL MAX

(d) Find open intervals on which the graph of f is concave up/down.

$$f''(x) = -12x - 6$$
$$= -6(2x+1) = 0$$
$$\Rightarrow x = -\frac{1}{2}$$

SIGNS OF f'': $+$ | $-$ at $-\frac{1}{2}$

CU | CD

GRAPH IS CU ON $(-\infty, -\frac{1}{2})$ AND CD ON $(-\frac{1}{2}, \infty)$

5. (8 points) Evaluate the indefinite integral: $\int \left(\sqrt{x} - x^4 + 5e^{-2x} + x^{-1}\right) dx$.

$$= \frac{2}{3}x^{3/2} - \frac{1}{5}x^5 + \frac{5}{-2}e^{-2x} + \ln|x| + C$$

6. (8 points) A grapefruit is tossed straight up with an initial velocity of 50 ft/sec. The grapefruit is 5 feet above the ground when it is released. Its height, in feet, at time t seconds is given by

$$h(t) = -16t^2 + 50t + 5.$$

Compute $h(2)$ and $h'(2)$. Using units, explain what each of these values represents.

$h(2) = -16(4) + 50(2) + 5 = 41$ FT $=$ HEIGHT AFTER 2 SEC

$h'(x) = -32t + 50$

$h'(2) = -14$ FT/SEC $\Rightarrow$ AFTER 2 SEC, GRAPEFRUIT IS FALLING AT 14 FT/SEC.

7. (10 points) Find the global (absolute) extreme values of the function $f(x) = x^4 - 2x^2$ on the interval $[-1, 2]$.

$f'(x) = 4x^3 - 4x$

$= 4x(x^2 - 1) = 0$

$x = 0,\ x = 1,\ x = -1$

ENDPTS: $x = -1$

$x = 2$

$f(-1) = -1$

$f(0) = 0$

$f(1) = -1$

$f(2) = 8$

$\boxed{\text{GLOBAL MIN IS } -1 \quad \text{GLOBAL MAX IS } 8}$

8. (10 points) Use the second derivative to determine whether the graph of $y = x^2 e^{-x}$ is concave up or down at the point where $x = 1$.

$$\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x}$$

$$\frac{d^2y}{dx^2} = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 2e^{-1} - 2e^{-1} - 2e^{-1} + e^{-1} = \underbrace{-e^{-1}}_{\approx -0.368} < 0 \Rightarrow \boxed{\text{CD AT } x = 1}$$

9. (8 points) Some values of the continuous function g are given in the table below.

x	0	0.5	1	1.5	2	2.5	3	3.5
$g(x)$	-0.10	1.4	2.9	2.8	2.3	1.5	-0.12	-0.93

(a) Find a reasonable estimate for $g'(3)$.

$$g'(3) \approx \frac{g(3.5) - g(2.5)}{3.5 - 2.5} = \frac{-0.93 - 1.5}{1} = \boxed{-2.43}$$

(b) Assume that g has a single critical point. Estimate the value of that critical point. Explain your reasoning.

IT LOOKS LIKE g IS INCREASING UP TO ABOUT $X = 1$

AND THEN DECREASING AFTER THAT.

I ESTIMATE THE CRIT PT IS $\boxed{X = 1}$

10. (8 points) Use a left sum with 4 subintervals (rectangles) of equal width to estimate $\int_0^1 \frac{2}{x^2+2}\,dx$.

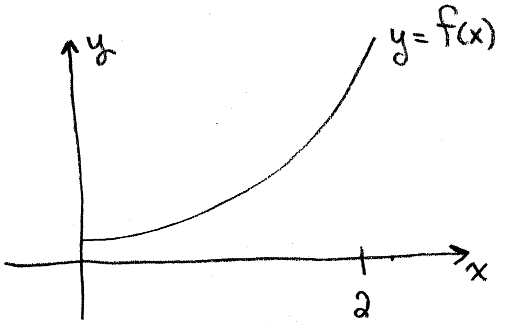
$$f(x) = \frac{2}{x^2+2} \qquad \Delta X = \frac{1-0}{4} = 0.25$$

PARTITION: $0 < 0.25 < 0.5 < 0.75 < 1$

$$\text{LEFT SUM} = 0.25\left[f(0) + f(0.25) + f(0.5) + f(0.75)\right]$$

$$\approx \boxed{0.90977}$$

11. (14 points) Use a definite integral (and the Fundmental Theorem of Calculus) to find the area of the region under the graph of $f(x) = 8x^3 + 6x^2 + 1$ over the interval from $x = 0$ to $x = 2$.



$$\int_0^2 (8x^3 + 6x^2 + 1)\,dx$$

$$= 2x^4 + 2x^3 + x \Big|_0^2$$

$$= (32 + 16 + 2) - (0 + 0 + 0)$$

$$= \boxed{50}$$

12. (8 points) Use a table of values to estimate the following limit. Your table must show function values at four or more points.

$$f(x) = \frac{1 - 7^x}{x}$$

$$\lim_{x \to 0} \frac{1 - 7^x}{x} \approx -1.95$$

x	f(x)
0.1	-2.14814
-0.1	-1.768287
0.01	-1.964966
-0.01	-1.9271
0.001	-1.947805
-0.001	-1.944018

13. (10 points) The revenue from selling q items is $R(q) = 900q - 2q^2$, and the total cost is $C(q) = 200 + 60q$. Find the quantity that maximizes profit. What is the total profit at that production level? Explain or show that you have indeed found a global maximum.

$$P(q) = 900q - 2q^2 - 200 - 60q$$

$$= -2q^2 + 840q - 200$$

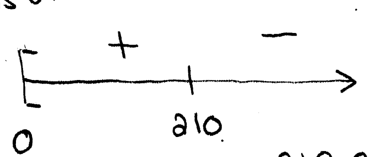
$$P'(q) = -4q + 840 = 0$$

$$\Rightarrow \boxed{q = 210}$$

MAX PROFIT IS

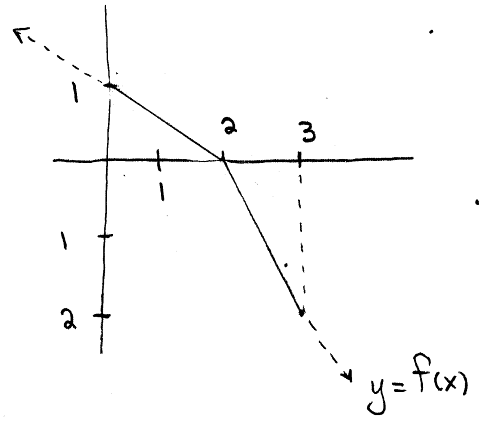
$$\boxed{P(210) = 88000}$$

SIGNS OF P'



q = 210 GIVES A MAX!

14. (6 points) Sketch the graph of a continuous function f with the properties that $\int_0^2 f(x)\,dx > 0$ but $\int_0^3 f(x)\,dx = 0$.



$$\int_0^2 f(x)\,dx = 1$$

$$\int_2^3 f(x)\,dx = -1$$

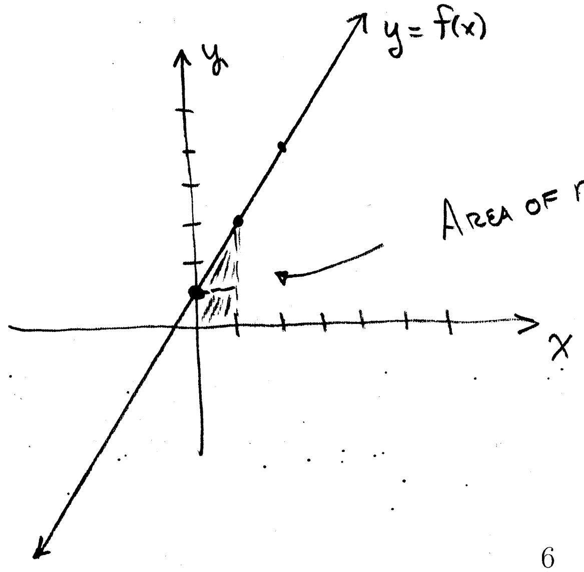
15. (8 points) The profit P (in dollars) for producing x units of a product is given by $P = -2x^2 + 72x - 145$. Compute the marginal profit at the level $x = 20$. Do you expect maximum profit to occur at a production level above or below 20 units? Explain your reasoning.

$$P'(x) = -4x + 72$$

$P'(20) = -8 \Rightarrow$ PROFIT IS DECREASING AT $X = 20$

MAX PROFIT IS BELOW $X = 20$

16. (8 points) Sketch the graph of $f(x) = 2x + 1$. Then use area to compute $\int_0^1 f(x)\,dx$.



AREA OF REGION IS

$$\frac{1}{2}(1)(2) + 1(1)$$

$$= \boxed{2}$$