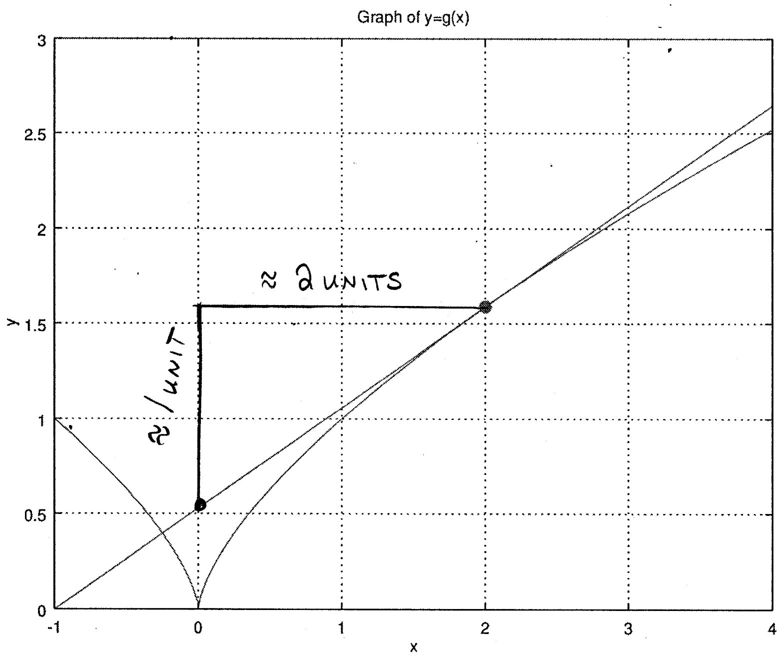
Math 171 - Test 2
October 17, 2018

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives, and do not simplify.

1. (6 points) The graph of $g(x) = \sqrt[3]{x^2}$ and its tangent line at $x = 2$ are shown below.



(a) Use the graph to estimate the slope of the tangent line.

$$m = \frac{\text{Rise}}{\text{Run}} \approx \frac{1}{2}$$

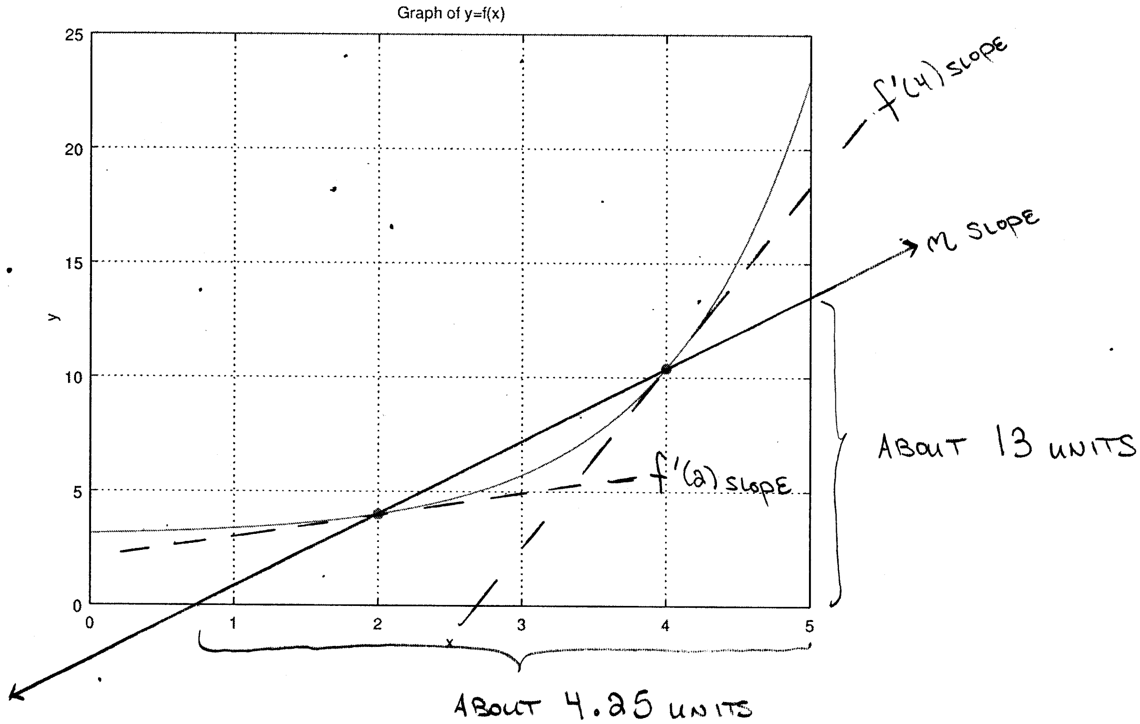
(b) Use the derivative to determine the slope of the tangent line.

$$g(x) = x^{2/3}$$

$$g'(x) = \frac{2}{3}x^{-1/3}$$

$$g'(2) = \frac{2}{3\sqrt[3]{2}} \approx 0.52913$$

2. (6 points) The graph of $y = f(x)$ is shown below.



(a) Sketch the secant line through the indicated points at $x = 2$ and $x = 4$. Let m be the slope of the secant line through those points. Estimate the value of m.

$$m = \frac{\text{RISE}}{\text{RUN}} \approx \frac{13}{4.25} \approx 3.$$

(b) Which number is greatest: m, $f'(2)$, or $f'(4)$?. Explain your reasoning.

$f'(4)$, TANGENT LINE AT $X=4$ IS STEEPEST. SEE ABOVE.

(c) Which number is least: m, $f'(2)$, or $f'(4)$?. Explain your reasoning.

$f'(2)$, TANGENT LINE AT $X=2$ IS LEAST STEEP. SEE ABOVE.

3. (6 points) Let $f(x) = 3x \sin x$. Find $f''(x)$.

$$f'(x) = 3 \sin x + 3x \cos x$$

$$f''(x) = 3 \cos x + 3 \cos x + 3x(-\sin x)$$

$$= \boxed{6 \cos x - 3x \sin x}$$

4. Suppose you launch an object straight upward with a velocity of 64 ft/sec from over the edge of the top of an 80-ft building. Use the position function

$$s(t) = -16t^2 + v_0 t + s_0,$$

where s represents height (in feet) at time t (in seconds), to solve the following problems.

(a) (1 point) Determine the function that gives the object's height at time t.

$$\boxed{S(t) = -16t^2 + 64t + 80}$$

(b) (2 points) Determine the function that gives the object's velocity at time t.

$$\boxed{S'(t) = -32t + 64}$$

(c) (2 points) Determine the object's velocity after 4 seconds. (Include units with your answer.)

$$S'(4) = -32(4) + 64 = \boxed{-64 \text{ FT/sec}}$$

(d) (3 points) Determine the object's maximum height. (Include units with your answer.)

$$S'(t) = 0 \Rightarrow -32t + 64 = 0$$

$$t = 2$$

$$S(2) = -16(4) + 64(2) + 80$$

$$= \boxed{144 \text{ FT}}$$

(e) (3 points) Determine when the object hits the ground.

$$S(t) = 0 \Rightarrow -16(t^2 - 4t - 5) = 0$$

$$-16(t-5)(t+1) = 0$$

$$\boxed{t = 5 \text{ sec}}$$

5. (20 points) Differentiate. Do not simplify.

(a) $\dfrac{d}{dx}\left(\dfrac{x^2}{2\sqrt{x}+1}\right)$

$$= \frac{(2\sqrt{x}+1)(2x) - (x^2)(x^{-1/2})}{(2\sqrt{x}+1)^2}$$

(b) $\dfrac{d}{dx}\left[(x^3-x)\sec x\right]$

$$= (x^3-x)\sec x \tan x + (3x^2-1)\sec x$$

(c) $\dfrac{d}{d\theta}(2\cos\theta - 5\sin\theta)^3$

$$= 3(2\cos\theta - 5\sin\theta)^2 \frac{d}{d\theta}(2\cos\theta - 5\sin\theta)$$

$$= 3(2\cos\theta - 5\sin\theta)^2(-2\sin\theta - 5\cos\theta)$$

(d) $\dfrac{d}{dt}\sin((\pi t+1)^2)$

$$= \cos((\pi t+1)^2)\frac{d}{dt}(\pi t+1)^2$$

$$= \cos((\pi t+1)^2)(2)(\pi t+1)(\pi)$$

$$= 2\pi(\pi t+1)\cos((\pi t+1)^2)$$

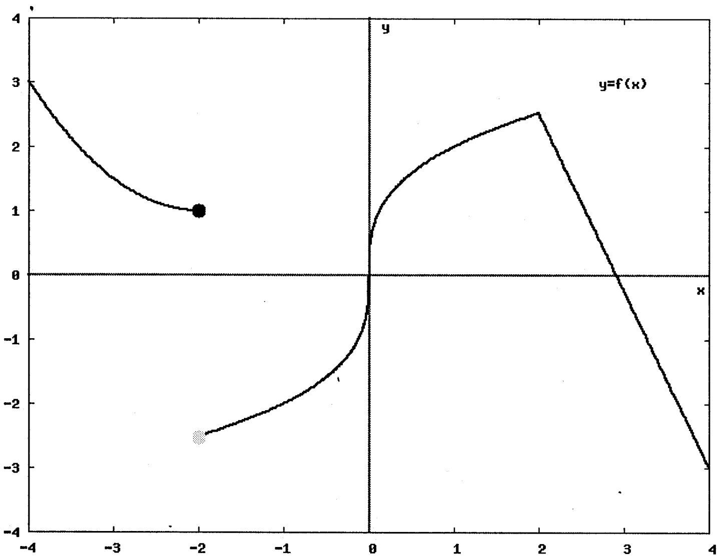
6. (8 points) Let $f(x) = x^2 - 2x + 5$. Use the **limit definition of the derivative** to find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^2 - 2(x+h) + 5] - [x^2 - 2x + 5]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h + \cancel{5} - \cancel{x^2} + \cancel{2x} - \cancel{5}}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h - 2)}{h} = \lim_{h \to 0} (2x + h - 2) = \boxed{2x - 2}$$

7. (3 points) The graph of $y = f(x)$ is shown below. Give the x-coordinates of three points at which $f'(x)$ does not exist. For each point, tell why f' does not exist.



X = -2 --- f'(2) DNE BECAUSE f IS DISCONTINUOUS AT THAT POINT

X = 0 --- f'(0) DNE BECAUSE THE TANGENT LINE AT X = 0 IS VERTICAL

X = 2 --- f'(2) DNE BECAUSE GRAPH HAS A SHARP POINT

8. (6 points) Find an equation of the line tangent to the graph of $y = \dfrac{x^3 - 3x^2 + 4}{x^2}$ at the point where $x = 1$.

$y = x - 3 + 4x^{-2}$

$\dfrac{dy}{dx} = 1 - 8x^{-3}$

$= 1 - \dfrac{8}{x^3}$.

Slope = $m = \dfrac{dy}{dx}\Big|_{x=1} = 1 - \dfrac{8}{1} = -7$

Point: $x = 1$, $y = 1 - 3 + 4 = 2$

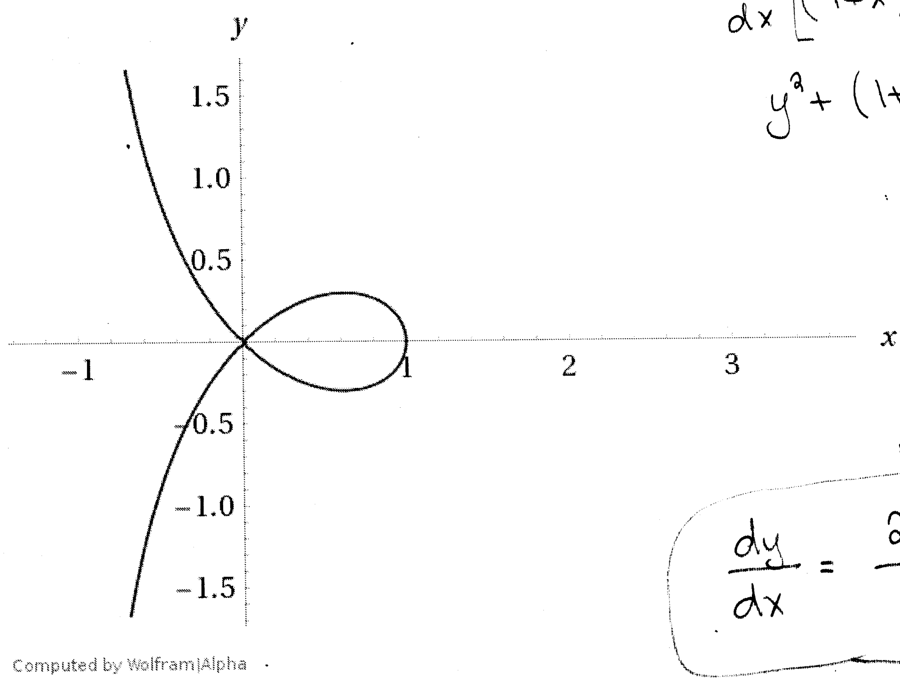
Tan line:

$y - 2 = -7(x-1)$

or

$y = -7x + 9$

9. (8 points) The graph shown below is called a *right strophoid*. It is the graph of the equation $(1+x)y^2 = (1-x)x^2$. Use implicit differentiation to find dy/dx.



$\dfrac{d}{dx}\left[(1+x)y^2\right] = \dfrac{d}{dx}(x^2 - x^3)$

$y^2 + (1+x)(2y)\dfrac{dy}{dx} = 2x - 3x^2$

$$\boxed{\frac{dy}{dx} = \frac{2x - 3x^2 - y^2}{2y(1+x)}}$$

10. (6 points) Determine each point (both coordinates) on the graph of $y = x^4 - 2x^2 + 3$ at which the tangent line is horizontal.

$$\frac{dy}{dx} = 4x^3 - 4x$$

TANGENT LINE HORIZONTAL $\Rightarrow$ $\frac{dy}{dx} = 0$

$$4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1) = 0$$

$$\Rightarrow x = 0,\ x = -1,\ x = 1$$

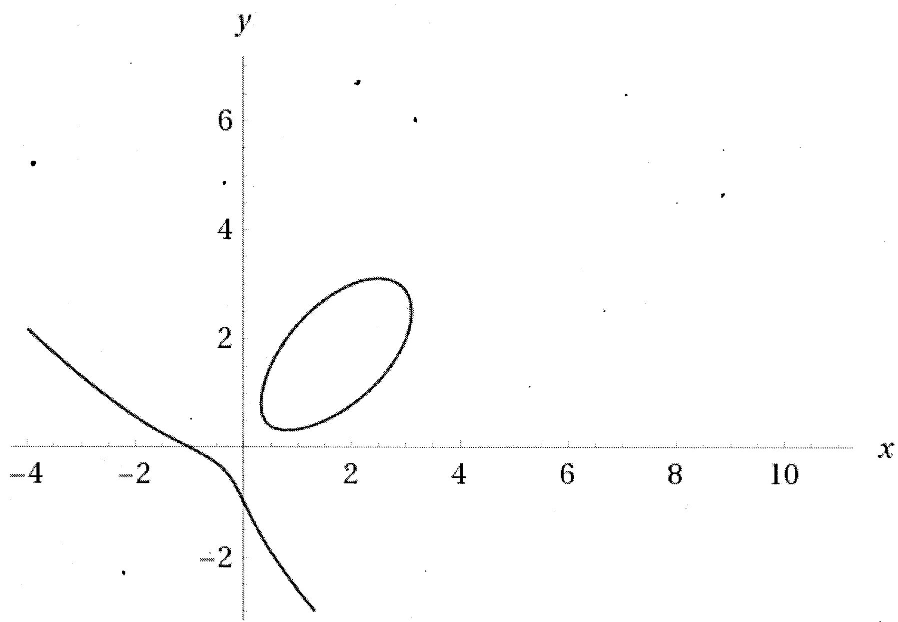
$x = 0,\ y = 3$ $\qquad$ $x = -1,\ y = 1 - 2 + 3 = 2$ $\qquad$ $x = 1,\ y = 1 - 2 + 3 = 2$

$(0,3),\ (-1,2),\ (1,2)$

Deliberately blank.

Take home problems–Due Monday, October 22.

11. (10 points) The graph of the equation $x^3 + y^3 = 6xy - 1$ is shown below. Find an equation of the tangent line at the point $(2, 3)$.



Computed by Wolfram|Alpha

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(6xy - 1\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = \frac{18-12}{27-12} = \frac{6}{15} = \frac{2}{5}$$

TANGENT LINE AT $(2,3)$:

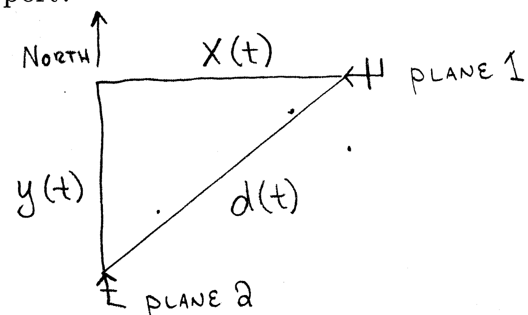
$$y - 3 = \frac{2}{5}(x - 2)$$

OR

$$y = \frac{2}{5}x + \frac{11}{5}$$

12. (10 points) Flying at the same constant elevation, two small planes approach an airport. One plane is flying due west at 120 mi/hr and the other is flying due north at 150 mi/hr. How fast is the distance between the planes changing at the moment when the westbound plane is 180 mi from the airport and the northbound plane is 225 mi from the airport?

Variables:



X = DISTANCE (IN MILES) FROM AIRPORT TO PLANE 1 AT TIME t

y = DISTANCE (MILES) FROM AIRPORT TO PLANE 2 AT TIME t

d = DISTANCE BETWEEN PLANES AT TIME t

t = TIME IN HOURS

Given information and what to find:

$\frac{dx}{dt} = -120$ FIND $\frac{dd}{dt}$ WHEN

$\frac{dy}{dt} = -150$ $X = 180$ & $y = 225$

Equation(s) relating the variables:

$$x^2 + y^2 = d^2$$

Equation(s) relating the rates:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2d\frac{dd}{dt} \quad \text{OR} \quad \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{d} = \frac{dd}{dt}$$

Solution: WHEN $x = 180$ & $y = 225$, $d = \sqrt{180^2 + 225^2} = \sqrt{83025} = 45\sqrt{41}$.

$$\frac{180(-120) + 225(-150)}{45\sqrt{41}} = -\frac{1230}{\sqrt{41}} \approx -192.1 \text{ mi/hr}$$

Follow-up question: Is your final answer negative or positive? Interpret the sign of your answer.

THE DISTANCE BETWEEN THE PLANES IS DECREASING.