Math 171 - Test 3

November 19, 2018

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Given a function f and its first and second derivatives, f' and f'', which would you use and how would you use it to determine

   (a) if f has a critical point at $x = 3$?

   Use f'. Check that $x=3$ is a domain interior pt. at which $f'(x)=0$ or $f'(x)$ DNE.

   (b) the zeros of f?

   Use f. Solve $f(x) = 0$.

   (c) if the graph of f has an inflection point at $x = -1$?

   Use f''. Check if the signs of f'' change at $x=-1$.

   (d) intervals on which f is decreasing?

   Use f'. $f'(x) < 0$ where f is decreasing.

2. (8 points) Find the absolute extreme values of $h(t) = \dfrac{2t}{t^2+1}$ on $[-3, 2]$.

$$h'(t) = \frac{(t^2+1)(2)-(2t)(2t)}{(t^2+1)^2} = \frac{2-2t^2}{(t^2+1)^2} = 0 \Rightarrow 2-2t^2=0$$

$$\Rightarrow t = \pm 1$$

Crit pts:
$t=-1,\ t=1$

End pts:
$t=-3,\ t=2$

$h(-3) = -\frac{3}{5}$

$h(2) = \frac{4}{5}$

$h(-1) = -1 \leftarrow$ Abs min

$h(1) = 1 \leftarrow$ Abs max

$f'(x) = \frac{f(3)-f(0)}{3-0}$ on $(0,3)$

3. (6 points) Let $f(x) = x^3 - 3x^2 + 8x$. Find a number that satisfies the conclusion of the Mean Value Theorem on the interval $[0,3]$.

$f'(x) = 3x^2 - 6x + 8$

$f(0) = 0$

$f(3) = 27 - 27 + 24 = 24$

$\frac{f(3)-f(0)}{3-0} = \frac{24}{3} = 8$

$f'(x) = 8$

$\Rightarrow 3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ OR $\boxed{x = 2}$

THIS IS THE ONLY SUCH NUMBER.

4. (6 points) Find the critical numbers of $g(x) = x^{2/3}(x^2 - 4)$.

$g'(x) = \frac{2}{3}x^{-1/3}(x^2-4) + x^{2/3}(2x)$

$= \frac{2(x^2-4)}{3x^{1/3}} + \frac{x^{2/3}(2x)}{1} \cdot \frac{3x^{1/3}}{3x^{1/3}}$

$= \frac{2x^2 - 8 + 6x^2}{3x^{1/3}} = \frac{8x^2-8}{3x^{1/3}}$

$g'(x) = 0$ WHEN

$8(x^2-1) = 0$

$\boxed{x = 1, -1}$

$g'(x)$ DNE WHEN

$\boxed{x = 0}$

5. (6 points) Let $r(x) = x^3 + \sin(10x)$. Without looking at the graph of r, determine whether the graph is concave up or down at the point where $x = 0.65$.

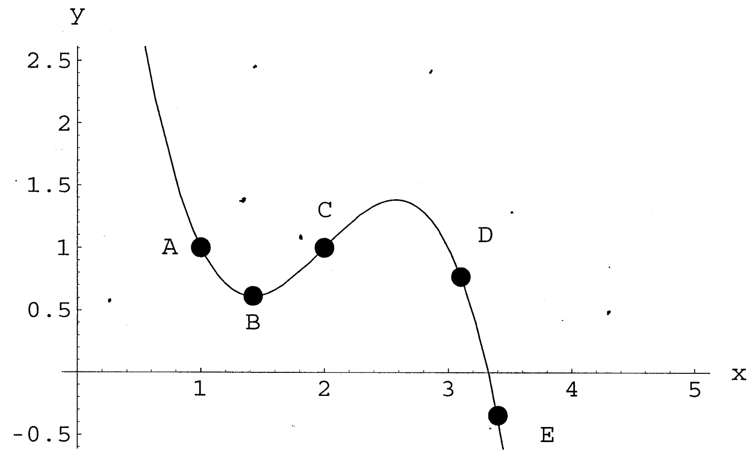
$r'(x) = 3x^2 + 10\cos 10x$

$r''(x) = 6x - 100 \sin 10x$

$r''(0.65) = 3.9 - 100\sin(6.5) \approx -17.612 < 0$

$\Rightarrow$ CONCAVE DOWN

6. (6 points) The graph of f is shown below. For each part of this problem, find a labeled point that satisfies the given condition.



(a) $f''(x) = 0$ C -- POINT OF INFLECTION

(b) $f'(x) = 0$ B -- HORIZONTAL TAN. LINE

(c) $f''(x) < 0$ D, E -- GRAPH IS CONCAVE DOWN

(d) $f(x) < 0$ E -- GRAPH BELOW X-AXIS

(e) $f'(x) > 0$ C -- f IS INCREASING AT C

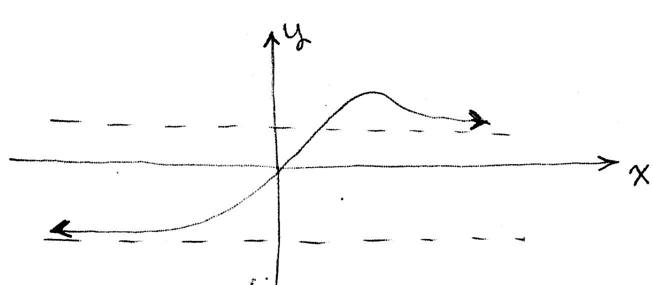
(f) $f''(x) > 0$ A, B -- GRAPH IS CONCAVE UP.

7. (3 points) Explain why Rolle's Theorem does not apply to $f(x) = \tan x$ on $[0, \pi]$.

f(x) IS NOT CONTINUOUS ON $[0, \pi]$ AND DIFFERENTIABLE ON $(0, \pi)$. IN FACT, f HAS AN INFINITE DISCONTINUITY AT $X = \pi/2$.

8. (4 points) What is the maximum number of horizontal asymptotes that the graph of a function can have? Sketch the graph of a function with that number of horizontal asymptotes.

$\boxed{2}$



9. (10 points) Let $f(x) = (x^2 - 4)^7$. Use the 1st derivative to find open intervals on which f is increasing/decreasing. Find all relative extreme values of f.

$$f'(x) = 7(x^2-4)^6(2x)$$

$$f'(x) = 0 \Rightarrow 2x = 0 \text{ OR } x^2-4=0$$

$$x = 0 \qquad x = \pm 2$$

SIGNS OF f'	$-$		$-$		$+$		$+$
		-2		0		2	
f IS	DEC		DEC		INC		INC

f IS DECREASING ON $(-\infty,-2) \cup (-2,0)$.

f IS INCREASING ON $(0,2) \cup (2,\infty)$.

$f(0) = (-4)^7 = -16384$ IS A REL. MIN

10. (10 points) Let $g(x) = \dfrac{1}{x^2+3}$. Find all inflection points of the graph of g.

$$g'(x) = \frac{(x^2+3)(0)-(1)(2x)}{(x^2+3)^2} = \frac{-2x}{(x^2+3)^2}$$

$$g''(x) = \frac{(x^2+3)^2(-2)-(-2x)(2)(x^2+3)(2x)}{(x^2+3)^4}$$

$$= \frac{-2(x^2+3)+(2x)(2)(2x)}{(x^2+3)^3}$$

$$= \frac{-6+6x^2}{(x^2+3)^3} = 0 \Rightarrow 6(x^2-1)=0$$

$$\Rightarrow x = \pm 1$$

$g''(x)$ DNE NOWHERE.

SIGNS OF g''	$+$		$-$		$+$
		-1		$+1$	
GRAPH IS	CU		CD		CU

INFLECTION PTS:

$\left(-1, \frac{1}{4}\right), \left(1, \frac{1}{4}\right)$

11. (6 points) Find the linearization of $f(x) = x^3 + \sqrt{x}$ at $x = 1$ and use it to approximate $(0.98)^3 + \sqrt{0.98}$.

$$f'(x) = 3x^2 + \frac{1}{2}x^{-1/2}$$

$$f'(1) = 3 + \frac{1}{2} = 3.5$$

$$f(1) = 1 + 1 = 2$$

$$\boxed{L(x) = 2 + 3.5(x-1)}$$

$$L(0.98) = 2 + 3.5(-0.02)$$

$$= 2 - 0.07 = 1.93$$

$$\boxed{f(1) \approx 1.93}$$

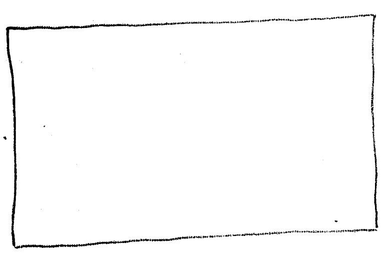
12. (4 points) Determine the following limit. Show work to justify your answer.

$$\lim_{x\to\infty} \frac{5x^4 - 8x^3 + 9x}{8x^7 - 99x^4} \cdot \frac{\frac{1}{x^7}}{\frac{1}{x^7}}$$

$$= \lim_{x\to\infty} \frac{\frac{5}{x^3} - \frac{8}{x^4} + \frac{9}{x^6}}{8 - \frac{99}{x^3}} = \frac{0}{8}$$

$$= \boxed{0}$$

13. (7 points) The length and width of a rectangle are reciprocals of one another. Find the minimum perimeter of such a rectangle.



WIDTH $= \frac{1}{x}$

LENGTH $= x$

$$\text{PERIMETER} = P(x) = 2x + \frac{2}{x}, \quad x > 0$$

$$P'(x) = 2 - \frac{2}{x^2} = 0$$

$$\Rightarrow x^2 = 1 \Rightarrow x = 1$$

SIGNS OF P': $-$ on $(0, 1)$, $+$ on $(1, \infty)$

P IS: DEC on $(0,1)$, INC on $(1,\infty)$

$\boxed{P(1) = 4 \text{ IS THE MINIMUM PERIMETER.}}$

14. (6 points) Determine the following limit. Show work to justify your answer.

$$\lim_{x\to\infty} \frac{\sqrt{8x^2+6x-5}}{5x-13} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\sqrt{8x^2+6x-5} \cdot \frac{1}{\sqrt{x^2}}}{(5x-13)\cdot \frac{1}{x}}$$

SINCE $\sqrt{x^2} = x$ FOR POSITIVE x.

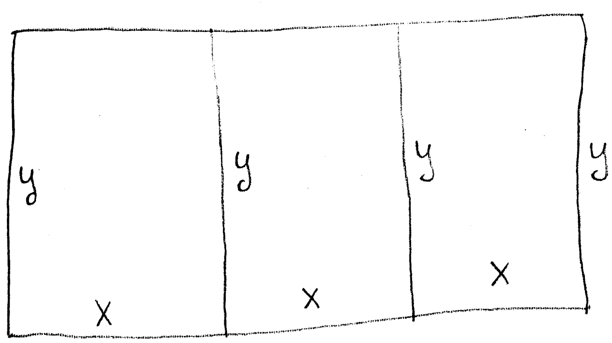
Is the limit any different as $x \to -\infty$? If so, explain.

$$\lim_{x\to\infty} \frac{\sqrt{8+\frac{6}{x}-\frac{5}{x^2}}}{5-\frac{13}{x}}$$

$$= \boxed{\frac{\sqrt{8}}{5}}$$

AS $x \to -\infty$, THE LIMIT IS $-\frac{\sqrt{8}}{5}$ SINCE $\sqrt{x^2} = -x$ FOR NEGATIVE x.

15. (10 points) A rancher has 1600 feet of fencing material that he will use to enclose three adjacent equal-size rectangular pens. Find the dimensions of the pens that maximize the combined area.



x = WIDTH OF SINGLE PEN

y = LENGTH OF PEN

$$6x + 4y = 1600$$

MAXIMIZE COMBINED AREA $= 3xy$.

$$y = \frac{1600-6x}{4} = 400 - \tfrac{3}{2}x$$

$$\text{AREA} = A(x) = 3x\left(400 - \tfrac{3}{2}x\right),$$

$$0 \le x \le \frac{1600}{6} = \frac{800}{3}$$

$$A(x) = 1200x - \tfrac{9}{2}x^2$$

$$A'(x) = 1200 - 9x = 0$$

$$\Rightarrow x = \frac{1200}{9} = \frac{400}{3}$$

$A(0) = 0$

$A\left(\frac{800}{3}\right) = 0$

$A\left(\frac{400}{3}\right) = (400)(200) = 80000$ ← ABS MAX.

DIMENSIONS FOR MAX AREA:

$x = \frac{400}{3}$ FT, $y = 200$ FT.