Math 171 - Final Exam

\#1

December 10, 2018

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) The graph of $y = f(x)$ is shown below, with dashed lines showing asymptotes. Use the graph to estimate each limit. Use $+\infty$, $-\infty$, or DNE if appropriate.

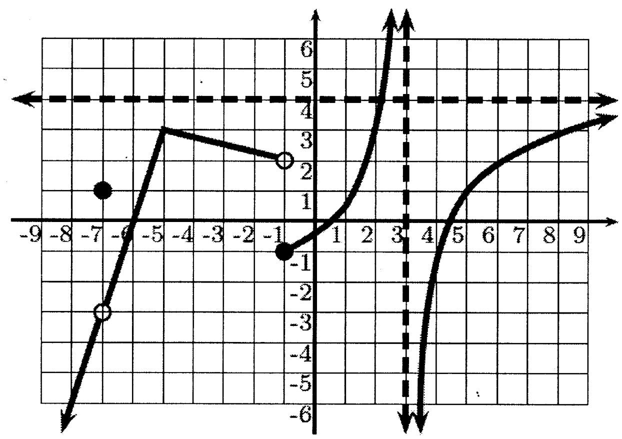
   (a) $\lim_{x\to -7} f(x) = -3$

   (b) $\lim_{x\to -1} f(x)$ DNE

   (c) $\lim_{x\to 5} f(x) = 1$

   (d) $\lim_{x\to 3^+} f(x) = -\infty$

   (e) $\lim_{x\to \infty} f(x) = 4$



2. (6 points) Refer to the graph of $y = f(x)$ shown above.

   (a) State each x-value at which f is discontinuous. For each discontinuity, tell whether it is removable or nonremovable.

   $X = -7$ REMOVABLE

   $X = -1$ NONREMOVABLE

   $X = 3$ NONREMOVABLE

   (b) State each x-value at which f is not differentiable.

   $X = -7$ $X = -1$

   $X = -5$ $X = 3$

3. (6 points) Evaluate the limit analytically: $\lim_{x\to 3} \frac{x(x-1)-6}{x^2-7x+12}$.

$\frac{0}{0}$ More work

$$\lim_{x\to 3} \frac{x^2-x-6}{x^2-7x+12} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x-4)}$$

$$= \frac{5}{-1} = \boxed{-5}$$

4. (10 points) Differentiate each function. Do not simplify.

(a) $f(x) = 8x^3(5x-2)^4$

$$f'(x) = 24x^2(5x-2)^4 + 32x^3(5x-2)^3(5)$$

(b) $g(x) = \frac{7\cos 3x}{\sqrt{x}}$

$$g'(x) = \frac{(\sqrt{x})(-21\sin 3x) - (7\cos 3x)(\frac{1}{2}x^{-1/2})}{x}$$

5. (8 points) Find an equation of the line tangent to the graph of $y = \frac{2}{\pi}x^2 + \tan x$ at the point where $x = \pi$.

Slope:

$$\frac{dy}{dx} = \frac{4}{\pi}x + \sec^2 x$$

$$\left.\frac{dy}{dx}\right|_{x=\pi} = 4 + \sec^2 \pi$$

$$= 5$$

Point: $x = \pi$

$$y = \frac{2}{\pi}(\pi)^2 + \tan \pi$$

$$= 2\pi$$

Tan Line:

$$y - 2\pi = 5(x-\pi) \text{ -or- }$$

$$y = 5x - 3\pi$$

6. (10 points) Assume that y is implicitly defined as a function of x by the equation $7x + x^2y = 4 + 3y^2$. Find dy/dx.

$$\frac{d}{dx}\left(7x + x^2y\right) = \frac{d}{dx}\left(4 + 3y^2\right)$$

$$7 + x^2 \cdot \frac{dy}{dx} + 2xy = 6y \cdot \frac{dy}{dx}$$

$$\left(x^2 - 6y\right)\frac{dy}{dx} = -2xy - 7$$

$$\frac{dy}{dx} = \frac{-2xy-7}{x^2-6y}$$

7. (18 points) Let $f(x) = x^3 - 9x^2 + 15x + 3$. Find open intervals on f is increasing/decreasing and determine the relative extreme values. Then find open intervals on which the graph of f is concave up/down and locate any points of inflection.

$$f'(x) = 3x^2 - 18x + 15$$
$$= 3(x^2 - 6x + 5)$$
$$= 3(x-5)(x-1) = 0$$
$$x = 5, \quad x = 1$$

SIGNS OF f': $+$ | 1 | $-$ | 5 | $+$

f IS: INC | 1 | DEC | 5 | INC

f IS INCREASING ON $(-\infty, 1) \cup (5, \infty)$

f IS DECREASING ON $(1, 5)$

$f(1) = 10$ IS A REL. MAX.

$f(5) = -22$ IS A REL. MIN.

$$f''(x) = 6x - 18 = 0$$
$$x = 3$$

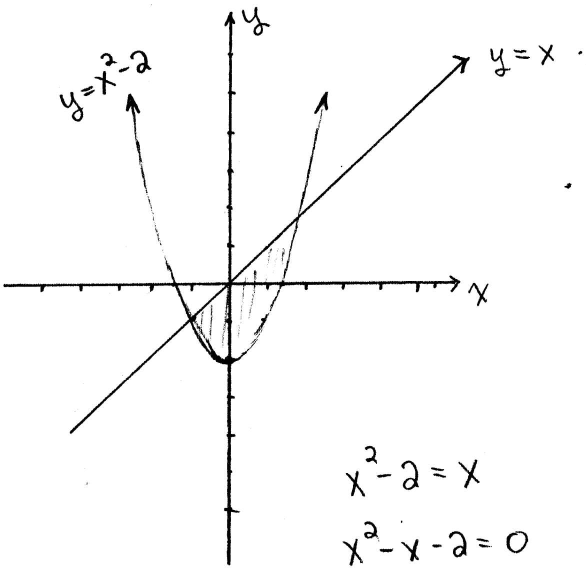
SIGNS OF f'': $-$ | 3 | $+$

GRAPH IS: CD | 3 | CU

GRAPH IS CONCAVE DOWN ON $(-\infty, 3)$ AND CONCAVE UP ON $(3, \infty)$

$(3, f(3)) = (3, -6)$ IS THE ONLY INFLECTION POINT.

8. (12 points) Find the area of the bounded region between the graphs of $y = x^2 - 2$ and $y = x$.



$x^2 - 2 = x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2,\ x = -1$

$$\int_{-1}^{2} x - (x^2 - 2)\, dx$$

$$= \int_{-1}^{2} (x - x^2 + 2)\, dx$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x \Big|_{-1}^{2}$$

$$= \left(2 - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right)$$

$$= \boxed{\frac{9}{2}}$$

9. (8 points) Evaluate the indefinite integral: $\int 6x(x^2 + 9)^3\, dx$.

$u = x^2 + 9$

$du = 2x\, dx$

$3\, du = 6x\, dx$

$$3 \int u^3\, du = \frac{3}{4} u^4 + C$$

$$= \boxed{\frac{3}{4}(x^2 + 9)^4 + C}$$

10. (8 points) Use the **limit definition of derivative** to determine $f'(x)$ if $f(x) = x^2 + x$. (You will receive points only if you use the limit definition.)

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + (x+h) - x^2 - x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h+1)}{h} = \lim_{h\to 0} (2x+h+1)$$

$$= \boxed{2x+1}$$

11. (12 points)

(a) Use four subintervals to compute a Riemann sum that approximates $\int_1^2 (x-1)\,dx$.

$\Delta x = \frac{2-1}{4} = \frac{1}{4} = 0.25$

$1 < 1.25 < 1.5 < 1.75 < 2$

Choose $c_1 = 1.2$, $c_2 = 1.4$, $c_3 = 1.6$, $c_4 = 1.8$

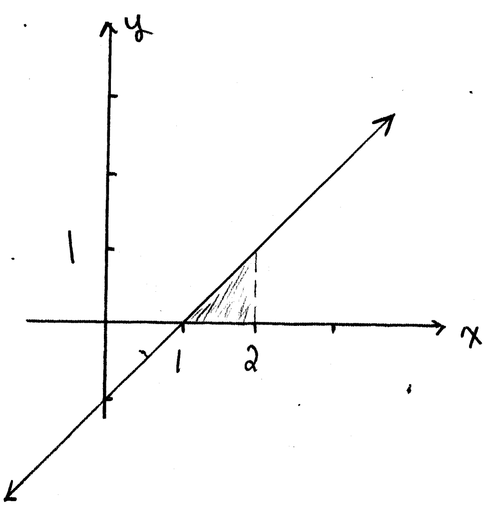
$f(x) = x - 1$

Riemann Sum $= \sum_{i=1}^{4} f(c_i)\Delta x$

$= 0.25(0.2 + 0.4 + 0.6 + 0.8)$

$= \boxed{0.5}$

(b) **Use area** to determine the exact value of $\int_1^2 (x-1)\,dx$.



Area of $\Delta = \frac{1}{2}(1)(1) = \boxed{\frac{1}{2}}$

The remaining problems all have multiple-choice solutions. Each problem is worth 3 points. **You must show your work to receive full credit.** The answer itself is worth 1 point. Your work/explanation is worth 2 points.

1. Use Newton's method staring with $x_0 = 2$ to approximate a solution of $x^2 = \cos x$. Which of these is closest to your value of x_2?

   (a) 1.10045

   (b) 0.82413

   (c) 0.85539

   (d) 0.73911

   $f(x) = x^2 - \cos x$

   $f'(x) = 2x + \sin x$

   $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $x_0 = 2$

   $x_1 = 1.10045$

   $x_2 = 0.85539$

2. Which one of the following is $\lim_{x\to5^+} \frac{7x}{x-5}$? (Show work or explain.)

   (a) ∞

   (b) $-\infty$

   (c) 0

   (d) 7

   $\frac{k \neq 0}{0}$ WITH SIGNS $\frac{+}{+}$

   $\Rightarrow +\infty$

3. The height of an object launched upward is given by $h(t) = -16t^2 + 64t + 80$, where $h(t)$ represents height (in feet) at time t (in seconds). What is the maximum height of the object?

   (a) 80 feet

   (b) 144 feet

   (c) 64 feet

   (d) 145 feet

   $h'(t) = -32t + 64 = 0$

   $t = 2$

   $h(2) = -16(4) + 64(2) + 80$

   $= 144$

4. Which of the following equations best relates the rates of change with respect to time t of the area A and the radius r of a circle?

   (a) $A = \pi r^2$

   (b) $\frac{dA}{dt} = \pi \frac{dr}{dt}$

   (c) $\frac{dA}{dt} = 2\pi \frac{dr}{dt}$

   (d) $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

   $A = \pi r^2$

   $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

GRAPH

CONCAVE UP

5. If f is a function with $f'(4) = 0$ and $f''(4) = 6$, then which one of the following must be true?

(a) $f(4)$ is a relative maximum.

(b) $f(4)$ is a relative minimum.

(c) The graph of f has an inflection point at $x = 4$.

(d) f is increasing at $x = 4$.

HOR. TAN. LINE.

FLAT SPOT

X=4 GIVES A MIN.

$\cdot \frac{1/x^2}{1/x^2}$

6. Which one of the following is $\lim_{x\to\infty} \frac{-10x+3}{5x^2+x+6}$? (Show work or explain.)

(a) ∞

(b) $-\infty$

(c) 0

(d) -2

$$\lim_{x\to\infty} \frac{-\frac{10}{x} + \frac{3}{x^2}}{5 + \frac{1}{x} + \frac{6}{x^2}} = \frac{0}{5}$$

7. Suppose f is a function with $f(1) = 2$ and $\lim_{x\to 1^-} f(x) = -\infty$. Which one of the following must be true?

(a) $x = 1$ is a vertical asymptote of the graph of f.

(b) $y = 2$ is a horizontal asymptote of the graph of f.

(c) $\lim_{x\to 1^+} f(x) = +\infty$

(d) f is differentiable at $x = 1$.

X=1 IS A V.A.

BY DEFINITION OF V.A.

8. Which one of the following best describes the line tangent to the graph of $f(x) = (3x - 12)^{1/3}$ at the point $(4, 0)$?

(a) The tangent line is horizontal.

(b) The tangent line is vertical.

(c) A unique tangent line does not exist.

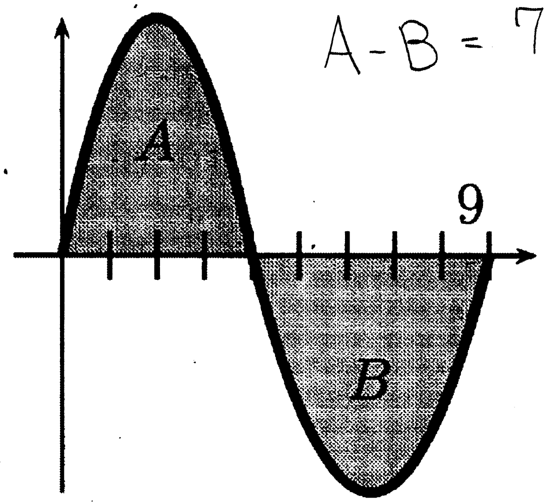
(d) The tangent line cannot be determined from the given information.

$$f'(x) = \frac{1}{3}(3x-12)^{-2/3}(3) = \frac{1}{\sqrt[3]{(3x-12)^2}}$$

HAS FORM $\frac{k \neq 0}{0}$ AT X=4

9. The graph of $y = f(x)$ is shown below. Region A has area 7, and region B has area 8. Which one of the following is the value of $\int_0^9 f(x)\,dx$?

$A - B = 7 - 8 = \boxed{-1}$



(a) 15

(b) 1

(c) -1

(d) 0

10. Use calculus techniques to find the absolute maximum value of $f(x) = 2x^3 + 3x^2 - 12x + 6$ on the interval $[-4, 2]$.

(a) -26

(b) -2

(c) 26

(d) 19

$f'(x) = 6x^2 + 6x - 12$

$= 6(x^2 + x - 2)$

$= 6(x+2)(x-1) = 0$

$x = -2, \ x = 1$

$f(-2) = 26 \qquad f(-4) = -26$

$f(1) = -1 \qquad f(2) = 10$

11. Use the Fundamental Theorem of Calculus to evaluate $\int_0^\pi \sin x\,dx$.

(a) 1

(b) 0

(c) 1.975

(d) 2

$\int_0^\pi \sin x \, dx$

$= -\cos x \Big|_0^\pi$

$= -\cos \pi + \cos 0$

$= -(-1) + 1 = 2$

12. Let $r(x) = x^3 + \sin(10x)$. Without looking at the graph of r, determine whether the graph is concave up or down at the point where $x = 0.63$.

(a) concave down

(b) concave up

(c) There is an inflection point at 0.63.

(d) r is not defined at $x = 0.63$.

$r'(x) = 3x^2 + 10\cos 10x$

$r''(x) = 6x - 100\sin 10x$

$r''(0.63) \approx 2.1$

GRAPH IS CU.

13. Use differentials to approximate the change in $f(x) = 2x^3 - \sqrt[3]{x}$ as x changes from $x = 1$ to $x = 1.1$.

(a) $\Delta y \approx 0.62972$

(b) $\Delta y \approx 1.62972$

(c) $\Delta y \approx 0.69472$

(d) $\Delta y \approx 0.56667$

$dy = f'(x)\,dx$

$\Delta y \approx \left(6x^2 - \frac{1}{3}x^{-2/3}\right)(\Delta x)$

$\Delta y \approx \left(6 - \frac{1}{3}\right)(0.1)$

$= 0.566\ldots$

14. Suppose you used a table of numerical values to approximate $\lim_{x\to 0} \dfrac{2^x - 1}{x}$. Which one is these is the most reasonable estimate for the limit?

$f(x) =$

(a) 0.70000

(b) 0.69555

(c) 0.69315

(d) The limit does not exist.

$X = 0.01$, $f(0.01) \approx 0.696$

$X = 0.0001$, $f(0.0001) \approx 0.693$

$X = 0.000001$, $f(0.000001)$

≈ 0.693