Math 172 - Test 1

September 21, 2016

Name ______________________

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Consider the function $f(x) = \ln\left(\frac{\sqrt{9+x^2}}{2x}\right)$.

   (a) Use logarithm laws to completely expand $f(x)$.

   $f(x) = \frac{1}{2}\ln(9+x^2) - \ln 2 - \ln x$

   (b) Determine $f'(x)$.

   $f'(x) = \frac{1}{2}\frac{1}{9+x^2}\cdot 2x - 0 - \frac{1}{x} = \frac{x}{9+x^2} - \frac{1}{x}$

   (c) Find the slope of the line tangent to the graph of f at the point where $x = 4$.

   $f'(4) = \frac{4}{25} - \frac{1}{4} = \frac{-9}{100}$

2. (8 points) Evaluate the definite integral: $\displaystyle\int_1^{e^2} \frac{(\ln x)^2}{x}\,dx$

   $u = \ln x$ $\qquad$ $x = 1 \Rightarrow u = 0$

   $du = \frac{1}{x}dx$ $\qquad$ $x = e^2 \Rightarrow u = 2$

   $\displaystyle\int_0^2 u^2\,du = \frac{u^3}{3}\Big|_0^2 = \frac{8}{3}$

3. (10 points) Let $g(x) = x^3 - \dfrac{4}{x}$.

(a) Explain how we can be certain that g has an inverse (as long as we stay away from $x = 0$).

$$g'(x) = 3x^2 + \frac{4}{x^2}$$

$g'(x) > 0$ FOR ALL $x \neq 0$.

$\Rightarrow$ $g(x)$ IS INCREASING $\Rightarrow$ g IS $1-1$ $\Rightarrow$ g HAS AN INVERSE.

(b) Compute $(g^{-1})'(6)$.

$$\frac{1}{g'(g^{-1}(6))}$$

$$g'(x) = 3x^2 + \frac{4}{x^2}$$

$$g^{-1}(6) = u \Leftrightarrow u^3 - \frac{4}{u} = 6$$

$$\Leftrightarrow u = 2$$

$$= \frac{1}{3(2)^2 + \frac{4}{2^2}} = \frac{1}{13}$$

4. (6 points) Use logarithmic differentiation to compute $f'(4)$.

$$f(x) = \frac{x(x-1)^3}{\sqrt{x+1}}$$

$$\ln f(x) = \ln x + 3 \ln(x-1) - \tfrac{1}{2} \ln(x+1)$$

$$\frac{1}{f(x)} f'(x) = \frac{1}{x} + \frac{3}{x-1} - \frac{1}{2}\frac{1}{x+1}$$

$$f'(4) = f(4)\left[\frac{1}{4} + \frac{3}{3} - \frac{1}{2}\frac{1}{5}\right] = \frac{108}{\sqrt{5}}\left(\frac{23}{20}\right) = \frac{621}{5\sqrt{5}}$$

$$\approx 55.54$$

5. (14 points) Find the relative extreme values of the function $f(x) = e^{12x-x^3}$. (Keep in mind that $e^x > 0$ for all x.)

$$f'(x) = e^{12x-x^3}\,(12-3x^2)$$

$f'(x)$ IS DEFINED EVERYWHERE

AND $f'(x) = 0$ IFF $12 - 3x^2 = 0$

$$x^2 = 4$$

$$x = \pm 2$$

SIGNS OF $f'(x)$: $-$ $\quad$ $+$ $\quad$ $-$

$x = -2$ $\quad$ $x = +2$

f IS DEC $\quad$ INC $\quad$ DEC

$f(-2) = e^{-16}$ IS A REL MIN

$f(2) = e^{16}$ IS A REL MAX

6. (9 points) Evaluate the definite integral: $\displaystyle\int_0^1 x\,3^{x^2}\,dx$

$u = x^2$ $\qquad$ $x = 0 \Rightarrow u = 0$

$du = 2x\,dx$ $\qquad$ $x = 1 \Rightarrow u = 1$

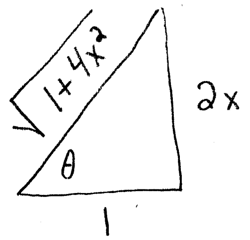
$$\frac{1}{2}\int_0^1 3^u\,du$$

$$= \frac{1}{2\ln 3}\,3^u\Big|_0^1 = \frac{1}{2\ln 3}(3-1)$$

$$= \frac{1}{\ln 3} \approx 0.91$$

7. (6 points) Use a right triangle to rewrite $\sec(\tan^{-1} 2x)$ as an algebraic expression.

$\theta = \text{TAN}^{-1}\ 2x \Rightarrow \text{TAN}\ \theta = \frac{2x}{1}$



$\sec\theta = \sqrt{1+4x^2}$

8. (8 points) Find the exact value of each expression. Show all work and/or explain your reasoning. (You may refer to your trig unit circle.)

(a) $\log_3 \frac{1}{81} = -4$ SINCE $3^{-4} = \frac{1}{81}$

(b) $e^{5\ln 2}e^{-\ln 4} = e^{\ln 2^5} e^{\ln 4^{-1}} = \frac{32}{4} = \boxed{8}$

(c) $\cos^{-1}\left(-\frac{1}{2}\right)$ = ANGLE IN $[0, \pi]$ WHOSE COS IS $-\frac{1}{2}$ $= \frac{2\pi}{3}$ OR $120°$

(d) $\underbrace{\tan^{-1}\left(\tan\frac{7\pi}{6}\right)}_{\text{TAN}\ \frac{7\pi}{6} = \text{TAN}\ \frac{\pi}{6}} = \text{TAN}^{-1}\left(\text{TAN}\ \frac{\pi}{6}\right) = \frac{\pi}{6}$

9. (8 points) Consider the function $f(x) = \tan^{-1}\sqrt{x}$. Find the coordinates of the point at which $f'(x) = 1/20$.

$$f'(x) = \frac{1}{1+(\sqrt{x})^2}\left(\frac{1}{2}x^{-1/2}\right) = \underbrace{\frac{1}{2\sqrt{x}(1+x)}} = \frac{1}{20}$$

$$\sqrt{x}(1+x) = 10$$

EASY TO OBSERVE

$$x = 4$$

$$(4, f(4)) = (4, \tan^{-1} 2) \approx (4, 1.107)$$

10. (14 points) Evaluate the indefinite integral: $\displaystyle\int \frac{2x-9}{\underbrace{x^2+6x+13}} dx$

$$x^2+6x+9+4$$

$$(x+3)^2+4$$

$$\int \frac{2x-9}{(x+3)^2+4} dx \qquad u = x+3 \qquad du = dx \qquad 2x-9 = 2(u-3)-9$$

$$= \int \frac{2u-15}{u^2+4} du = \int \frac{2u}{u^2+4} du - \int \frac{15}{u^2+4} du$$

$$w = u^2+4$$

$$dw = 2u\,du$$

$$\int \frac{1}{w} dw = \ln|w|$$

$$= \ln(u^2+4) - \frac{15}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \ln((x+3)^2+4) - \frac{15}{2}\tan^{-1}\frac{x+3}{2} + C$$

11. (7 points) Evaluate the indefinite integral: $\displaystyle\int \frac{\cos 2t}{3+\sin 2t}\,dt$

$u = 3 + \sin 2t$

$du = 2\cos 2t \, dt$

$\frac{1}{2} du = \cos 2t \, dt$

$$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|3+\sin 2t| + C$$

12. (2 points) True or False: If the inverse of f exists, then the y-intercept of the graph of f is the x-intercept of the graph of f^{-1}.

True

$f \qquad f^{-1}$

$(a,b) \longleftrightarrow (b,a)$