# Math 172 - Test 3a

November 23, 2016

Name key

Score ______

Show all work. Supply explanations where necessary. There should be no need for a calculator.

1. (8 points) Evaluate the limit: $\lim_{x\to 0} \frac{x^3}{\sin x - x}$. $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{3x^2}{\cos x - 1} \quad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{6x}{-\sin x} \quad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{6}{-\cos x} = -6$$

2. (8 points) Explain why the integral $\int_0^\infty e^{-x}\,dx$ is improper. Write it as a limit of proper integrals and evaluate.

THE INTEGRAL IS IMPROPER BECAUSE THE INTEGRATION INTERVAL IS UNBOUNDED $([0,\infty))$.

$$\lim_{c\to\infty} \int_0^c e^{-x}\,dx = \lim_{c\to\infty} -e^{-x}\Big|_0^c$$

$$= \lim_{c\to\infty} e^{-x}\Big|_c^0 = e^0 - \lim_{c\to\infty} e^{-c}$$

$$= 1 - 0 = \boxed{1}$$

3. (8 points) Use the partial fraction decomposition to evaluate $\displaystyle\int \underbrace{\frac{5x}{x^3 - x^2 + 4x - 4}}_{x^2(x-1)+4(x-1)} \, dx$.

$$= (x^2+4)(x-1)$$

$$\frac{5x}{(x^2+4)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$5x = A(x^2+4) + (Bx+C)(x-1)$$

$x=1$: $5 = 5A \Rightarrow A=1$

$x=0$: $0 = 4A - C \Rightarrow C=4$

$x=-1$: $-5 = 5A + (-B+C)(-2)$

$$-5 = 5 + 2B - 8 \Rightarrow B=-1$$

$u = x^2+4$

$\frac{1}{2}du = x\,dx$

$$\int \left[\frac{1}{x-1} + \frac{-x+4}{x^2+4}\right] dx = \int \frac{1}{x-1}\,dx - \int \frac{x}{x^2+4}\,dx + 4\int \frac{1}{x^2+4}\,dx$$

$$= \ln|x-1| - \frac{1}{2}\ln(x^2+4) + 2\tan^{-1}\frac{x}{2} + C$$

4. (8 points) Integrate: $\displaystyle\int_0^{\pi/2} 7\sin^3\theta\cos^4\theta\,d\theta$ $= \displaystyle\int_0^{\pi/2} 7\left(1-\cos^2\theta\right)\cos^4\theta\sin\theta\,d\theta$

$u = \cos\theta \qquad \theta = 0 \rightarrow u = 1$

$du = -\sin\theta\,d\theta \qquad \theta = \frac{\pi}{2} \rightarrow u = 0$

$$-\int_1^0 7\left(1-u^2\right)u^4\,du$$

$$= 7\int_0^1 \left(u^4-u^6\right)du = 7\left(\frac{1}{5}-\frac{1}{7}\right)$$

$$= \frac{14}{35} = \boxed{\frac{2}{5}}$$

5. (8 points) Integrate: $\displaystyle\int 5x^3e^{2x}\,dx$. $= \boxed{\left(\frac{5}{2}x^3 - \frac{15}{4}x^2 + \frac{30}{8}x - \frac{30}{16}\right)e^{2x} + C}$

(Hint: Use the tabular method.)

| Signs | u & derivs | dv/dx and antis |
|---|---|---|
| + | $5x^3$ | $e^{2x}$ |
| − | $15x^2$ | $\frac{1}{2}e^{2x}$ |
| + | $30x$ | $\frac{1}{4}e^{2x}$ |
| − | $30$ | $\frac{1}{8}e^{2x}$ |
| + | $0$ | $\frac{1}{16}e^{2x}$ |

# Math 172 - Test 3b

November 23, 2016

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary. YOU MUST WORK INDIVIDUALLY ON THIS EXAM.

1. (7 points) Evaluate the definite integral $\int_{\pi/6}^{\pi/3} \underbrace{\sin 6x \cos 4x}\, dx$. (Using a product-to-sum formula is probably easiest.)

$$\sin 6x \cos 4x = \frac{1}{2}(\sin 10x + \sin 2x)$$

$$\int_{\pi/6}^{\pi/3} \left(\frac{1}{2}\sin 10x + \frac{1}{2}\sin 2x\right) dx$$

$$= -\frac{1}{20}\cos 10x - \frac{1}{4}\cos 2x \Big|_{\pi/6}^{\pi/3}$$

$$= -\frac{1}{20}\cos\frac{10\pi}{3} - \frac{1}{4}\cos\frac{2\pi}{3} + \frac{1}{20}\cos\frac{10\pi}{6} + \frac{1}{4}\cos\frac{2\pi}{6}$$

$$= -\frac{1}{20}\left(-\frac{1}{2}\right) - \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{20}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \frac{6}{20} = \frac{3}{10}$$

2. (8 points) Evaluate the integral $\int \tan^{-1} 2x\, dx$.

$w = 2x$

$\frac{1}{2}dw = dx$

$$\frac{1}{2}\int \tan^{-1} w\, dw = \frac{1}{2} w \tan^{-1} w - \underbrace{\frac{1}{2}\int \frac{w}{1+w^2}\, dw}$$

$u = \tan^{-1} w \qquad du = \frac{1}{1+w^2}\, dw$

$dv = 1\, dw \qquad v = w$

$y = 1 + w^2$

$\frac{1}{2}dy = w\, dw$

$\frac{1}{4}\int \frac{1}{y}\, dy$

$\frac{1}{4}\ln|y| + C$

$$\int \tan^{-1} 2x\, dx = \frac{1}{2}\int \tan^{-1} w\, dw$$

$$= \frac{1}{2} w \tan^{-1} w - \frac{1}{4}\ln(1+w^2) + C$$

$$= x \tan^{-1} 2x - \frac{1}{4}\ln(1+4x^2) + C$$

$0 \cdot \infty$

3. (5 points) Evaluate the limit: $\lim\limits_{x \to 0+} x^3 \ln x^2$.

$$\underbrace{\lim_{x \to 0^+} \frac{2 \ln x}{\frac{1}{x^3}} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to 0^+} \frac{\frac{2}{x}}{\frac{-3}{x^4}}}_{\text{L'Hôpital's Rule}} = \lim_{x \to 0^+} \frac{2x^3}{-3} = 0$$

4. (10 points) Consider the definite integral $\int_0^1 x^3 \ln x^2 \, dx$.

(a) Even though the integrand is not defined at $x = 0$, the integral is NOT improper (by our definition). Explain why not. (You'll need to refer to problem [illegible].)
#3

THE DISCONTINUITY AT $X = 0$ IS NOT AN INFINITE DISCONTINUITY (SEE PROB #3).

(b) Even though the integral is not improper, the lower bound should still be considered as a limit to zero from above. Evaluate the integral. (You will encounter an indeterminate form.)

$$\lim_{c \to 0^+} \int_c^1 x^3 \ln x^2 \, dx = \lim_{c \to 0^+} \left[ \frac{1}{4} x^4 \ln x^2 \Big|_c^1 - \int_c^1 \frac{1}{2} x^3 \, dx \right]$$

$u = \ln x^2 \qquad du = \frac{2}{x} dx$

$dv = x^3 dx \qquad v = \frac{1}{4} x^4$

$$= \lim_{c \to 0^+} \left[ \frac{1}{4} x^4 \ln x^2 - \frac{1}{8} x^4 \right]_{x=c}^{x=1}$$

$$= -\frac{1}{8} - \lim_{c \to 0^+} \left( \frac{1}{4} c^4 \ln c^2 - \frac{1}{8} c^4 \right)$$

$$= -\frac{1}{8} - \lim_{c \to 0^+} \left( \overset{\to 0}{\frac{1}{4} c} \cdot \overset{\to 0}{\underset{\text{See \#3}}{c^3 \ln c^2}} - \overset{\to 0}{\frac{1}{8} c^4} \right)$$

$$= \boxed{-\frac{1}{8}}$$

5. (15 points) Evaluate the definite integral $\int_4^8 \frac{\sqrt{x^2-16}}{x^2}\,dx$.

$x = 4\sec\theta$ $\qquad$ $x = 4 \Rightarrow \sec\theta = 1 \Rightarrow \theta = 0$

$dx = 4\sec\theta\tan\theta\,d\theta$ $\qquad$ $x = 8 \Rightarrow \sec\theta = 2 \Rightarrow \theta = \frac{\pi}{3}$

$$\int_0^{\pi/3} \frac{\sqrt{16\sec^2\theta - 16}}{16\sec^2\theta}\, 4\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3} \frac{|4\tan\theta|}{4\sec\theta}\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int_0^{\pi/3} \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$

$$= \int_0^{\pi/3} (\sec\theta - \cos\theta)\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta \Big|_0^{\pi/3}$$

$$= \left[\ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}\right] - \left[\ln(1) - 0\right]$$

$$= \boxed{\ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}}$$

6. (7 points) Evaluate the improper integral $\int_2^\infty \frac{1}{x \ln x}\, dx.$ $= \lim_{c \to \infty} \int_2^c \frac{1}{x \ln x} dx$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

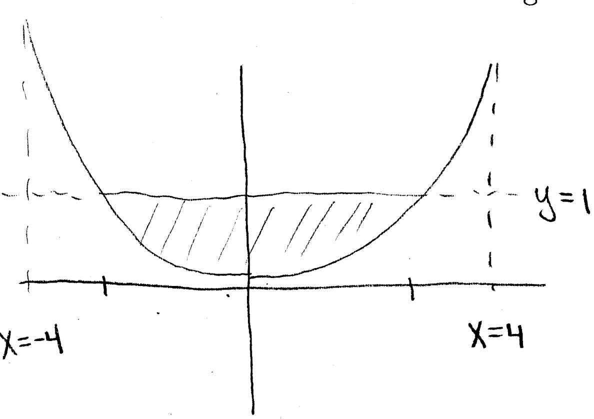
$$\lim_{c \to \infty} \int_{\ln 2}^{\ln c} \frac{1}{u} du = \lim_{c \to \infty} \ln u \Big|_{\ln 2}^{\ln c}$$

$$= \lim_{c \to \infty} \left[ \underbrace{\ln(\ln c)}_{\to \infty} - \ln(\ln 2) \right]$$

$$= +\infty$$

Diverges

7. (8 points) Find the area of the region bounded by the graphs of $y = 1$ and $y = \frac{7}{16 - x^2}$. Use a PFD for the integral.




$$1 = \frac{7}{16 - x^2}$$

$$\Downarrow$$

$$16 - x^2 = 7$$

$$\Downarrow$$

$$x^2 = 9$$

$$x = \pm 3$$

$$\int_{-3}^{3} \left(1 - \frac{7}{16 - x^2}\right) dx$$

$$\frac{7}{16 - x^2} = \frac{A}{4 + x} + \frac{B}{4 - x}$$

$$7 = A(4 - x) + B(4 + x)$$

$x = 4$: $7 = 8B$ $\quad B = 7/8$

$x = -4$: $7 = 8A$ $\quad A = 7/8$

$$\int_{-3}^{3} 1 - \frac{7}{16 - x^2} dx = 2 \int_0^3 1 - \frac{7}{16 - x^2} dx$$

$$= 2 \int_0^3 \left(1 - \frac{7/8}{4 + x} - \frac{7/8}{4 - x}\right) dx = 2 \left[ x - \frac{7}{8} \ln(4 + x) + \frac{7}{8} \ln(4 - x) \right]_0^3$$

8

$$= 2\left[3 - \frac{7}{8} \ln 7\right] - 2\left[0 - \frac{7}{8} \ln 4 + \frac{7}{8} \ln 4\right] =$$

$6 - \frac{7}{4} \ln 7 \approx 2.59$