# Math 172 - Final Exam A

December 7, 2016

Name Key

Score

Show all work to receive full credit. Supply explanations where necessary. YOU MUST WORK INDIVIDUALLY ON THIS EXAM.

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$.

   (a) (2 points) Without giving a rigorous proof, use the "looks like" idea to argue that the series converges.

"LOOKS LIKE" $\sum \frac{1}{4n^2} = \frac{1}{4} \sum \frac{1}{n^2}$

WHICH IS A CONVERGENT $p = 2$ SERIES.

$\therefore$ "THINK" THE ORIGINAL SERIES CONVERGES.

   (b) (6 points) Find the sum of the series. (Hint: Write as a telescoping series.)

$$\frac{1}{(2n+1)(2n+3)} = \frac{A}{2n+1} + \frac{B}{2n+3}$$

$$1 = A(2n+3) + B(2n+1)$$

$n = -\frac{3}{2}$ : $1 = -2B \Rightarrow B = -\frac{1}{2}$

$n = -\frac{1}{2}$ : $1 = 2A \Rightarrow A = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$$

$$= \frac{1}{2}\left[\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \cdots\right]$$

$$= \boxed{\frac{1}{6}}$$

2. (18 points) Determine whether each series converges conditionally, converges absolutely, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

$a_n = \frac{1}{\sqrt[3]{n}}$

TERMS CLEARLY DECREASE WITH $\lim_{n\to\infty} a_n = 0$ → CONVERGES BY AST.

HOWEVER, $\sum \frac{1}{\sqrt[3]{n}}$ IS A DIVERGENT $p = \frac{1}{3}$ SERIES. DOES NOT CONVERGE ABSOLUTELY

SERIES CONVERGES CONDITIONALLY.

(b) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$ DIVERGES BY $n^{TH}$ TERM TEST.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n 3^n}{n!}$

RATIO TEST...

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+1)3^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^{n+1} n\, 3^n} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)3}{(n+1)n} = \lim_{n\to\infty} \frac{3}{n} = 0$$

SERIES CONVERGES ABSOLUTELY BY RATIO TEST.

3. (8 points) Find the fourth Maclaurin polynomial for $f(x) = \ln(x^2+1)$. Then use it to approximate $\ln(1.04)$.

$f(x) = \ln(x^2+1)$; $f(0) = 0$

$f'(x) = \dfrac{2x}{x^2+1}$; $f'(0) = 0$

$f''(x) = \dfrac{2(x^2+1) - 4x^2}{(x^2+1)^2}$

$= \dfrac{2-2x^2}{(x^2+1)^2}$; $f''(0) = 2$

$f'''(x) = \dfrac{-4x(x^2+1)^2 - (2-2x^2)(2)(x^2+1)(2x)}{(x^2+1)^4}$

$= \dfrac{-4x^3 - 4x - 8x + 8x^3}{(x^2+1)^3} = \dfrac{4x^3 - 12x}{(x^2+1)^3}$; $f'''(0) = 0$

$f^{(4)}(x) = \dfrac{(12x^2-12)(x^2+1)^3 - (4x^3-12x)(3)(x^2+1)^2(2x)}{(x^2+1)^6}$

$= \dfrac{(12x^2-12)(x^2+1) - (4x^3-12x)(6x)}{(x^2+1)^4}$

$f^{(4)}(0) = -12$

$P_4(x) = \frac{2}{2!}(x-0)^2 - \frac{12}{4!}(x-0)^4$

$P_4(x) = x^2 - \frac{1}{2}x^4$

$\ln(1.04) \approx P_4(0.2)$

$\approx 0.0392$

4. (6 points) Does the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{(2n)!}$ converge conditionally or absolutely? Show work for full credit.

RATIO TEST:

$\lim_{n\to\infty} \left| \dfrac{(-1)^{n+1}}{(2n+2)!} \cdot \dfrac{(2n)!}{(-1)^n} \right|$

$= \lim_{n\to\infty} \dfrac{1}{(2n+2)(2n+1)} = 0 < 1$

CONVERGES ABSOLUTELY.

Follow up: How many terms are required to approximate the sum with an error of 0.0001 or less?

SERIES THE SERIES ALTERNATES, WE CAN USE AST REMAINDER THEOREM.

ERROR AFTER $n$ TERMS $\leq \dfrac{1}{a_{n+1}} = \dfrac{1}{(2n+2)!} < 0.0001$

$n = 3$ MAKES $\dfrac{1}{a_{n+1}} \approx 0.0000248 < 0.0001$

USE 3 TERMS.

5. (6 points) Evaluate: $\displaystyle\int_{-2}^{2} \frac{6}{(x-1)^3}\,dx$

$$\lim_{c\to 1^-} \int_{-2}^{c} \frac{6}{(x-1)^3}\,dx + \lim_{c\to 1^+} \int_{c}^{2} \frac{6}{(x-1)^3}\,dx$$

$$\lim_{c\to 1^-} \left.\frac{-3}{(x-1)^2}\right|_{-2}^{c} + \lim_{c\to 1^+} \left.\frac{-3}{(x-1)^2}\right|_{c}^{2}$$

$$\lim_{c\to 1^-} \left(\frac{-3}{(c-1)^2} + \frac{3}{9}\right) + \lim_{c\to 1^+} \left(\frac{-3}{1} + \frac{3}{(c-1)^2}\right)$$

$-\infty + \frac{3}{9}$ AND NO NEED TO COMPUTE FURTHER.

INTEGRAL DIVERGES!

6. (4 points) Evaluate: $\displaystyle\lim_{x\to 0+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

$\infty - \infty$

$$\lim_{x\to 0^+} \frac{(e^x-1)-x}{x(e^x-1)} \qquad 0/0$$

$$= \lim_{x\to 0^+} \frac{e^x-1}{xe^x + e^x - 1} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{e^x}{xe^x + e^x + e^x} = \boxed{\frac{1}{2}}$$

# Math 172 - Final Exam B

December 12, 2016

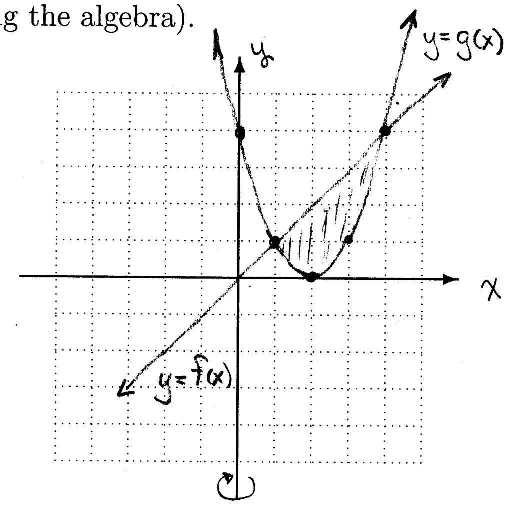
Name key

Score ________

**Show all work to receive full credit**. Supply explanations where necessary. All integrals must be evaluated by hand unless otherwise stated.

1. Consider the region between the graphs of the functions $f(x) = x$ and $g(x) = (x-2)^2$.

   (a) (4 points) Sketch the graphs of $f$ and $g$, identify the region, and find the $x$-bounds on the region (by doing the algebra).




$$x = x^2 - 4x + 4$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$
$$x = 1,\ x = 4$$

   (b) (7 points) Set up the two arc-length integrals required to find the perimeter of the region. Use your calculator to evaluate the integrals and find the perimeter.

$$\text{Perimeter} = \int_1^4 \sqrt{1+[f'(x)]^2}\,dx + \int_1^4 \sqrt{1+[g'(x)]^2}\,dx$$
$$= \int_1^4 \sqrt{2}\,dx + \int_1^4 \sqrt{1+(2(x-2))^2}\,dx$$
$$\approx 3\sqrt{2} + 6.1257 \approx 10.37$$

   (c) (7 points) The region is rotated about the $y$-axis. Set up the integral that gives the volume of the solid that is generated. Use your calculator to evaluate the integral.

Shells...

$$2\pi \int_1^4 x\left(x-(x-2)^2\right)dx \approx 70.69$$

2. (8 points) Use integration by parts to evaluate the following definite integral.

$$\int_0^{\pi} x \sin 2x \, dx$$

| | | |
|---|---|---|
| $+$ | $x$ | $\sin 2x$ |
| $-$ | $1$ | $-\frac{1}{2}\cos 2x$ |
| $+$ | $0$ | $-\frac{1}{4}\sin 2x$ |

$$= -\frac{1}{2}x\cos 2x \Big|_0^{\pi} + \frac{1}{4}\sin 2x \Big|_0^{\pi}$$

$$= \boxed{-\frac{\pi}{2}}$$

3. (5 points) Evaluate the limit: $\displaystyle \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$ $\quad \frac{\infty}{\infty}$

L'Hôpital's Rule

$$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-1/x^2\right)}} = 1$$

$$\therefore \boxed{\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} = 1}$$

4. (5 points) Evaluate the indefinite integral: $\displaystyle \int \frac{e^{\sqrt{x}}}{5\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$$\frac{2}{5}\int e^u \, du$$

$$= \frac{2}{5}e^u + C$$

$$= \boxed{\frac{2}{5}e^{\sqrt{x}} + C}$$

$f'(x) = -\frac{1}{x^2}$

5. Let $f(x) = \dfrac{1}{x} + 3$. In this problem, you will find $(f^{-1})'(1)$ using two different approaches.

(a) (5 points) Use our formula for the derivative of an inverse function to find $(f^{-1})'(1)$.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{-\frac{1}{(-\frac{1}{2})^2}} = -\frac{1}{4}$$

$f^{-1}(1) = x$

$\Leftrightarrow \frac{1}{x} + 3 = 1$

$\Leftrightarrow x = -\frac{1}{2}$

(b) (5 points) Find the actual inverse function and then compute its derivative at $x = 1$.

$$y = \frac{1}{x} + 3 \Rightarrow \frac{1}{x} = y - 3 \Rightarrow x = \frac{1}{y-3} \Rightarrow f^{-1}(x) = \frac{1}{x-3}$$

$$(f^{-1})'(x) = \frac{-1}{(x-3)^2} \qquad (f^{-1})'(1) = \frac{-1}{(-2)^2} = -\frac{1}{4}$$

6. (12 points) Use a partial fraction decomposition to evaluate the following indefinite integral.

$$\int \frac{3x^2+5x+3}{\underbrace{x^3+x}_{x(x^2+1)}}\,dx$$

$$\frac{3x^2+5x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{3}{x} + \frac{5}{x^2+1}$$

$$3x^2+5x+3 = A(x^2+1) + (Bx+C)x$$
$$= (A+B)x^2 + Cx + A$$

$$\left.\begin{array}{l} A+B=3 \\ C=5 \\ A=3 \end{array}\right\} \quad B=0$$

$$\int \left(\frac{3}{x} + \frac{5}{x^2+1}\right) dx$$

$$= 3\ln x + 5\tan^{-1}x + C$$

7. (10 points) Use an appropriate trigonometric substitution to evaluate the following definite integral.

$$\int_0^2 \frac{1}{(x^2+4)^2}\,dx$$

$X = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$X = 2 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$X = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

$$\int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta + 4)^2} = \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta}$$

$$= \frac{1}{8}\int_0^{\pi/4} \cos^2\theta\, d\theta$$

Power reducing...

$$\frac{1}{16}\int_0^{\pi/4} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta \Big|_0^{\pi/4}$$

$$= \boxed{\frac{\pi}{64} + \frac{1}{32}}$$

8. (6 points) Write the following improper integral as a limit of proper integrals and evaluate.

$$\int_1^\infty \frac{2}{x^2}\,dx$$

$$\lim_{c\to\infty} \int_1^c 2x^{-2}\,dx$$

$$= \lim_{c\to\infty} -2x^{-1}\Big|_1^c = \lim_{c\to\infty} -\frac{2}{x}\Big|_1^c$$

$$= \lim_{c\to\infty} \frac{2}{x}\Big|_c^1$$

$$= 2 - \lim_{c\to\infty}\frac{2}{c}$$

$$= 2 - 0 = \boxed{2}$$

9. (4 points) Find the slope of the line tangent to the graph of $y = \sin^{-1}(e^{2x})$ at the point where $x = -1$.

$$\frac{dy}{dx} = \frac{2e^{2x}}{\sqrt{1-e^{4x}}} \qquad \left.\frac{dy}{dx}\right|_{x=-1} = \boxed{\frac{2e^{-2}}{\sqrt{1-e^{-4}}} \approx 0.2732}$$

10. (7 points) A thin plate covers the region in Problem 1. The density of the thin plate at the point $(x, y)$ is given by $\delta(x) = x$. Set up the integral that gives the plate's moment about the $x$-axis. Use your calculator to evaluate the integral.

$$M_x = \int_1^4 y\,dm = \int_1^4 \left(\frac{(x-2)^2+x}{2}\right)(x)\left(x-(x-2)^2\right)dx$$

$$= \frac{801}{40} = 20.025$$

$dm = \rho(x) \cdot \text{HEIGHT} \cdot dx$

11. (15 points) Determine whether each <u>series</u> converges or diverges. Show all work and explain your reasoning.

(a) $\sum_{n=0}^{\infty} \tan^{-1} n$

$$\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2} \Rightarrow$$ SERIES DIVERGES

BY $n^{\text{TH}}$ TERM TEST

(b) $\sum_{n=2}^{\infty} \frac{8}{7^n} = 8\sum_{n=2}^{\infty} \left(\frac{1}{7}\right)^n$

GEOMETRIC WITH $r = \frac{1}{7} < 1$

SERIES CONVERGES.

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right)$

$+ \cdots$

THIS SERIES IS TELESCOPING.

$$S_n = 1 - \frac{1}{2n+1} \longrightarrow 1 \text{ AS } n \to \infty.$$

SERIES CONVERGES WITH SUM 1.