Math 173 - Test 1

February 18, 2010

Name key

Score

Show all work. Supply explanations when necessary.

1. (10 points) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^3}$.

RATIO TEST:

$$\lim_{N\to\infty} \frac{|x+2|^{N+1}}{(N+1)^3} \cdot \frac{N^3}{|x+2|^N} = \lim_{N\to\infty} |x+2| \frac{N^3}{(N+1)^3}$$

$$= |x+2| \lim_{N\to\infty} \frac{N^3}{(N+1)^3} = |x+2| < 1$$

$$\Rightarrow -1 < x+2 < 1$$

$$-3 < x < -1$$

INTERVAL OF CONVERGENCE IS $[-3,-1]$

ENDPOINTS:

$x=-3$: $\sum_{N=1}^{\infty} \frac{(-1)^N}{N^3}$

$x=-1$: $\sum_{N=1}^{\infty} \frac{1^N}{N^3}$

$p=3$ SERIES. THEY CONVERGE.

2. (4 points) Suppose f is defined by the power series above.

$$f(x) = \sum_{n=1}^{\infty} \frac{(x+2)^n}{n^3}$$

Find a power series for $f'(x)$ and write out its first 5 terms.

$$f(x) = \sum_{N=1}^{\infty} \frac{(x+2)^N}{N^3} = (x+2) + \frac{(x+2)^2}{2^3} + \frac{(x+2)^3}{3^3} + \frac{(x+2)^4}{4^3} + \cdots$$

$$f'(x) = \sum_{N=1}^{\infty} \frac{(x+2)^{N-1}}{N^2} = 1 + \frac{(x+2)}{2^2} + \frac{(x+2)^2}{3^2} + \frac{(x+2)^3}{4^2} + \frac{(x+2)^4}{5^2} + \cdots$$

3. (4 points) Find a 2D vector of length 5 that makes an angle of $-\pi/6$ with the positive x-axis.

$$5\left(\cos\left(-\frac{\pi}{6}\right)\hat{\imath} + \sin\left(-\frac{\pi}{6}\right)\hat{\jmath} \right)$$

$$= 5\cos\frac{\pi}{6}\,\hat{\imath} - 5\sin\frac{\pi}{6}\,\hat{\jmath}$$

$$= \boxed{\frac{5\sqrt{3}}{2}\hat{\imath} - \frac{5}{2}\hat{\jmath}}$$

4. (6 points) What does it mean for two vectors to be orthogonal? Find a nonzero vector orthogonal to $\vec{u} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$.

TWO VECTORS ARE ORTHOGONAL IF THEIR DOT PRODUCT IS ZERO.

LET $\boxed{\vec{v} = 2\hat{\imath} + 2\hat{\jmath} - 2\hat{k}}$ THEN

$$\vec{u}\cdot\vec{v} = 6 - 4 - 2 = 0$$

5. (6 points) Find a vector of magnitude 3 that is parallel to the line segment joining $P(1, -2, 7)$ and $Q(9, 0, 3)$.

$$\vec{PQ} = (9-1)\hat{\imath} + (0+2)\hat{\jmath} + (3-7)\hat{k} = 8\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$$

$$|\vec{PQ}| = \sqrt{64+4+16} = \sqrt{84}$$

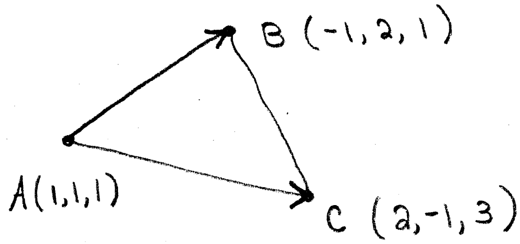
$$\boxed{\frac{3}{\sqrt{84}}\left(8\hat{\imath} + 2\hat{\jmath} - 4\hat{k}\right)} \quad \text{OR} \quad \frac{12}{\sqrt{21}}\hat{\imath} + \frac{3}{\sqrt{21}}\hat{\jmath} - \frac{6}{\sqrt{21}}\hat{k}$$

6. (4 points) Find the midpoint of the line segment $\overline{PQ}$ in the problem above.

$$\text{MIDPOINT} = \left(\frac{1+9}{2}, \frac{0+(-2)}{2}, \frac{3+7}{2}\right)$$

$$= \boxed{(5, -1, 5)}$$

7. (8 points) Find the area of the triangle with vertices $A(1,1,1)$, $B(-1,2,1)$, and $C(2,-1,3)$.



$$\vec{AB} = -2\hat{i} + \hat{j}$$

$$\vec{AC} = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 0 \\ 1 & -2 & 2 \end{vmatrix} =$$

$$\text{AREA} = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| \qquad = 2\hat{i} + 4\hat{j} + 3k$$

$$\text{AREA} = \frac{\sqrt{4+16+9}}{2} = \boxed{\frac{\sqrt{29}}{2}}$$

8. (8 points) Find an equation of the plane passing through the point $(1,7,3)$ and normal to the line with the following symmetric equations:

$$x = \frac{y-1}{3} = \frac{z}{4} \quad \Rightarrow \quad \frac{X-0}{1} = \frac{y-1}{3} = \frac{z-0}{4}$$

$$\vec{N} = \hat{i} + 3\hat{j} + 4\hat{k}$$

POINT $(1,7,3)$

$$1(x-1) + 3(y-7) + 4(z-3) = 0$$

$$\boxed{x + 3y + 4z = 34}$$

9. (8 points) Verify that the point $(1,-2,3)$ is not on the plane $6x - 7y + 3z = 18$. Then find the distance from the point to the plane.

VERIFY: $6(1) - 7(-2) + 3(3) = 6 + 14 + 9 = 29 \neq 18$

$Q\ (1,-2,3) \leftarrow$ GIVEN POINT

$P\ (3,0,0) \leftarrow$ POINT ON PLANE

$\vec{PQ} = -2\hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{N} = 6\hat{i} - 7\hat{j} + 3\hat{k} \leftarrow$ PLANE'S NORMAL

$$\text{DISTANCE} = \left| \text{Proj}_{\vec{N}} \vec{PQ} \right|$$

$$= \left| \frac{\vec{PQ} \cdot \vec{N}}{\vec{N} \cdot \vec{N}} \right| |\vec{N}| = \frac{|\vec{PQ} \cdot \vec{N}|}{|\vec{N}|}$$

$$= \frac{|-12 + 14 + 9|}{\sqrt{36 + 49 + 9}}$$

$$= \boxed{\frac{11}{\sqrt{94}}}$$

10. (18 points) Suppose $\vec{u} = 3\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{v} = -\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$. Find each of the following.

(a) $\vec{u} - \vec{v}$ $= (3+1)\hat{i} + (2+3)\hat{j} + (-1-2)\hat{k}$

$$= \boxed{4\hat{i} + 5\hat{j} - 3\hat{k}}$$

(b) The angle between $\vec{u}$ and $\vec{v}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{-3+(-6)+(-2)}{\sqrt{14}\,\sqrt{14}} = \frac{-11}{14}$$

$$\theta = \cos^{-1}\left(-\frac{11}{14}\right) = 2.475 = 141.79^\circ$$

(c) $\vec{u} \times \vec{u}$ $= \vec{0}$

IF VECTORS ARE PARALLEL, THEIR CROSS PRODUCT IS THE ZERO VECTOR.

(d) $\text{proj}_{\vec{u}}\vec{v}$ $= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{-11}{14}(3\hat{i} + 2\hat{j} - \hat{k}) = -\frac{33}{14}\hat{i} - \frac{22}{14}\hat{j} + \frac{11}{14}\hat{k}$

From (b)

(e) A unit vector in the direction of $\vec{v}$.

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{\sqrt{14}} = -\frac{1}{\sqrt{14}}\hat{i} - \frac{3}{\sqrt{14}}\hat{j} + \frac{2}{\sqrt{14}}\hat{k}$$

(f) The work done by the force $\vec{v}$ in displacing an object along the vector $\vec{u}$.

$$W = \vec{v}\cdot\vec{u} = \boxed{-11}$$

From (b)

11. (10 points) Consider the surface described by the equation $x^2 + \frac{y^2}{4} - z = 0$.

(a) Identify the surface.

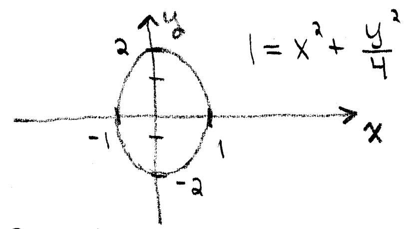
$$Z = X^2 + \frac{y^2}{4}$$

THE SURFACE IS A PARABOLOID W/ VERTEX AT (0,0,0) OPENING UP THE Z-AXIS.

(b) Describe (or sketch) in detail the level curve obtained by fixing $z = 1$.

$$1 = X^2 + \frac{y^2}{4}$$

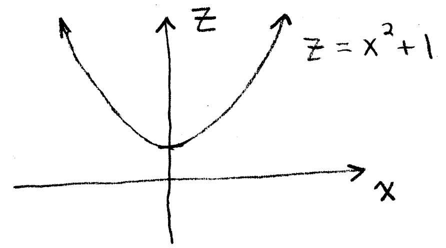
ELLIPSE CENTERED AT (0,0) WITH INTERCEPTS (0,±2) & (±1,0)



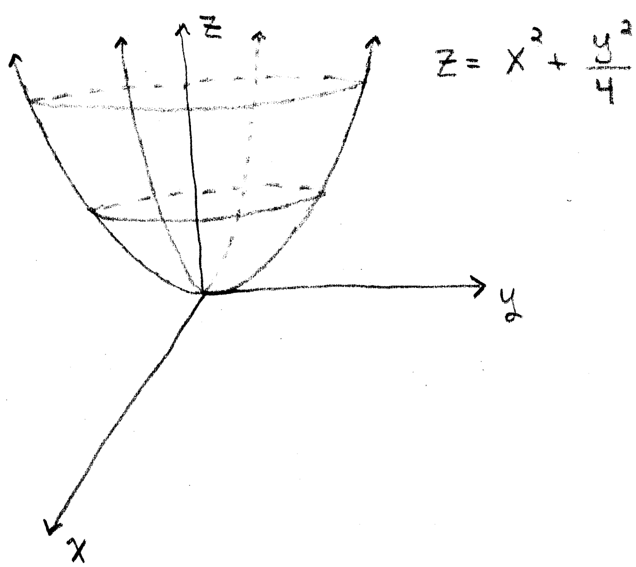
(c) Describe (or sketch) in detail the level curve obtained by fixing $y = 2$.

$$Z = X^2 + 1$$

STANDARD PARABOLA $Z = X^2$ SHIFTED UP 1 UNIT.



(d) Sketch a rough graph of the surface.



$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

12. (8 points) Find the 2nd Maclaurin polynomial for $f(x) = e^x \cos x$. Then use your polynomial to approximate $f(1)$.

$f(x) = e^x \cos x$, $f(0) = 1$

$f'(x) = e^x \cos x - e^x \sin x$ $f'(0) = 1$

$f''(x) = e^x \cos x - e^x \sin x$
$- (e^x \sin x + e^x \cos x)$ $f''(0) = 0$

$\boxed{P_2(x) = 1 + x}$

$\boxed{f(1) \approx P_2(1) = 2}$

13. (6 points) The Maclaurin series for $g(x) = \sin x$ is given by

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Use this series to find the Maclaurin series for $\frac{\sin x}{x}$ and explain how the new series could be used to establish the limit $\lim_{x\to\infty} \frac{\sin x}{x} = 1$.

Typo! SHOULD BE $x \to 0$

MAC SERIES FOR $\frac{\sin x}{x}$ IS $\boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots}$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \cdots\right)$$

$$= 1$$