Math 173 - Test 2a

March 18, 2010

Name key

Score

Show all work. Supply explanations when necessary.

1. (12 points) A golf ball leaves the ground at an angle of 30° with a speed of 90 ft/sec.

   (a) Find the vector-valued function $\vec{r}(t)$ that describes the motion of the golf ball. (Use $g = 32\,\text{ft/sec}^2$.)

   $$\vec{r}(t) = (90\cos 30^\circ)t\,\hat{i} + \left[-16t^2 + (90\sin 30^\circ)t\right]\hat{j}$$

   $$\boxed{\vec{r}(t) = 45\sqrt{3}\,t\,\hat{i} + (-16t^2 + 45t)\,\hat{j}}$$

   (b) Find the speed of the golf ball one second after it leaves the ground.

   $$\vec{r}\,'(t) = 45\sqrt{3}\,\hat{i} + (-32t + 45)\,\hat{j}$$

   $$\vec{r}\,'(1) = 45\sqrt{3}\,\hat{i} + 13\hat{j} \qquad |\vec{r}\,'(1)| = \sqrt{(45\sqrt{3})^2 + (13)^2}$$

   $$\approx \boxed{79.02\ \text{FT/SEC}}$$

   (c) Will the golf ball clear the top of a 30-ft tree 135 ft away?

   WHEN $\underbrace{45\sqrt{3}\,t = 135}$, WHAT IS THE HEIGHT?

   $$t = \frac{135}{45\sqrt{3}} = \sqrt{3} \qquad -16(\sqrt{3})^2 + 45\sqrt{3} = \boxed{29.942\ldots\ \text{FT}}$$

   NOT GONNA MAKE IT.

2. (5 points) Given the function $f(x, y) = e^x + x^2 \ln y + y^3 \ln x$, determine f_{xyx}.

   $$f_x(x,y) = e^x + 2x\ln y + \frac{y^3}{x}$$

   $$f_{xy}(x,y) = \frac{2x}{y} + \frac{3y^2}{x}$$

   $$\boxed{f_{xyx}(x,y) = \frac{2}{y} - \frac{3y^2}{x^2}}$$

3. (12 points) Determine each limit or explain why the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(4,1)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ $= \dfrac{16-4}{\sqrt{4}-\sqrt{1}} = \dfrac{12}{1} = \boxed{12}$

(b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x-y}{x+y-2}$ DNE.

ALONG $y=1$:

$\displaystyle\lim_{x\to 1} \frac{x-1}{x-1} = 1$

ALONG $y=x$:

$\displaystyle\lim_{x\to 1} \frac{0}{2x-2} = 0$

DIFFERENT LIMITS ALONG DIFFERENT PATHS.

$\Rightarrow$ $\boxed{\text{LIMIT DNE}}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ $= 0$

CONVERT TO POLAR...

$\displaystyle\lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta = \boxed{0}$

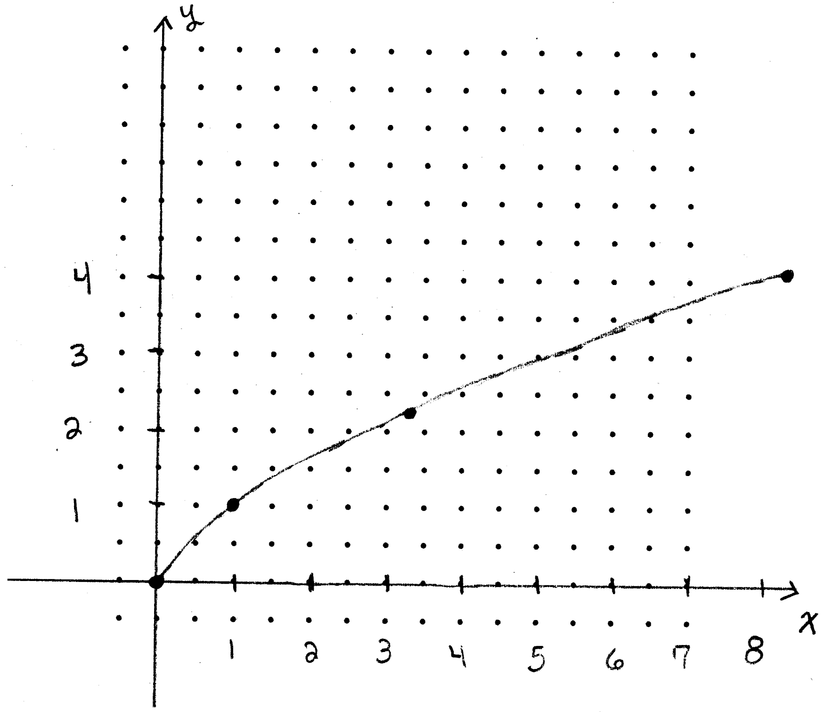
4. (2 points) Describe the motion of a particle if its normal component of acceleration is always zero.

THE PARTICLE CAN NEVER CHANGE DIRECTION.

IT IS EITHER NOT MOVING OR MOVING ALONG A LINE.

5. (8 points) Sketch the curve described by the vector-valued function and set up the definite integral that gives its length. Use your calculator to find or approximate the value of the integral.

$$\vec{r}(t) = t^3\hat{\imath} + t^2\hat{\jmath}, \qquad 0 \le t \le 2$$



t	$\vec{r}(t)$
0	$0\hat{\imath} + 0\hat{\jmath}$
$\frac{1}{2}$	$\frac{1}{8}\hat{\imath} + \frac{1}{4}\hat{\jmath}$
1	$\hat{\imath} + \hat{\jmath}$
$\frac{3}{2}$	$\frac{27}{8}\hat{\imath} + \frac{9}{4}\hat{\jmath}$
2	$8\hat{\imath} + 4\hat{\jmath}$

$$\int_0^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt \approx 9.0734$$

6. (7 points) Find all values of x and y where $f_x(x, y) = 0$ and $f_y(x, y) = 0$ simultaneously.

$$f(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$$

$$f_x(x,y) = 2x + 4y - 4 = 0$$

$$f_y(x,y) = 4x + 2y + 16 = 0$$

$$2x + 4y = 4$$
$$4x + 2y = -16$$

$\rightarrow$

$$-x - 2y = -2$$
$$4x + 2y = -16$$
$$3x = -18$$
$$x = -6$$

$$6 - 2y = -2$$
$$\Rightarrow y = 4$$

$$\boxed{(x,y) = (-6,4)}$$

7. (10 points) A curve is described by the vector-valued function $\vec{r}(t) = t\hat{\imath} + \sin t\hat{\jmath}$. Find the curvature at the point $(\pi/2, 1)$.

$t = \pi/2$

$\vec{r}\,'(t) = \hat{\imath} + \cos t\,\hat{\jmath}$

$\vec{r}\,'(\frac{\pi}{2}) = \hat{\imath}$

$|\vec{r}\,'(\frac{\pi}{2})| = 1$

$\hat{T}(t) = \frac{1}{\sqrt{1+\cos^2 t}}(\hat{\imath} + \cos t\,\hat{\jmath})$

$\frac{d\hat{T}}{dt} = -\frac{1}{2}(1+\cos^2 t)^{-3/2}(-2\cos t\sin t)(\hat{\imath} + \cos t\,\hat{\jmath}) + \frac{1}{\sqrt{1+\cos^2 t}}(-\sin t\,\hat{\jmath})$

$\left.\frac{d\hat{T}}{dt}\right|_{t=\pi/2} = -\hat{\jmath} \Rightarrow \left|\frac{d\hat{T}}{dt}\right| = 1$

$K = \frac{1}{|\vec{v}|}\left|\frac{d\hat{T}}{dt}\right| = \boxed{1}$

8. (8 points) Suppose $\vec{r}\,'(t) = \frac{1}{1+t^2}\hat{\imath} + \frac{1}{t}\hat{\jmath} + \frac{1}{\sqrt{t}}\hat{k}$.

$\mathbb{R}$ $\quad t \neq 0 \quad t > 0$

(a) What is the domain of $\vec{r}\,'$?

$\boxed{\{t : t > 0\}}$

(b) Find $\vec{r}$ so that $\vec{r}(1) = 2\hat{\imath} + \hat{\jmath}$.

$\boxed{\vec{r}(t) = (\tan^{-1} t + C_1)\hat{\imath} + (\ln t + C_2)\hat{\jmath} + (2\sqrt{t} + C_3)\hat{k}}$

$\vec{r}(1) = 2\hat{\imath} + \hat{\jmath} = (\frac{\pi}{4} + C_1)\hat{\imath} + C_2\hat{\jmath} + (2 + C_3)\hat{k} \Rightarrow$

$C_1 = 2 - \frac{\pi}{4}$

$C_2 = 1$

$C_3 = -2$

$(8, \pi/4, 3\pi/4)$

9. (3 points) Convert the point $(\rho, \theta, \phi) =$ ~~$(8, \pi/4, 3\pi/4)$~~ to rectangular coordinates.

$x = \rho \sin\phi \cos\theta \qquad x = 8 \sin\frac{3\pi}{4}\cos\frac{\pi}{4} = 4$

$y = \rho \sin\phi \sin\theta \qquad y = 8 \sin\frac{3\pi}{4}\sin\frac{\pi}{4} = 4$

$z = \rho \cos\phi \qquad z = 8\cos\frac{3\pi}{4} = -4\sqrt{2}$

$\boxed{(x,y,z) = (4, 4, -4\sqrt{2})}$

10. (12 points) Consider the function $g(x, y) = \ln(x^2 + y^2)$.

(a) What is the domain of g?

$$\{(x,y) : x^2+y^2 > 0\}$$

or $\{(x,y) : (x,y) \neq (0,0)\}$

(b) What is the range of g?

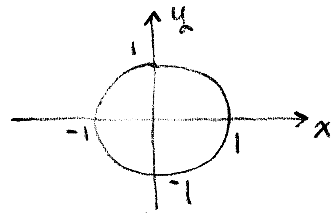
ANY NUMBER IS A POSSIBLE OUTPUT.

$$\text{RANGE} = \mathbb{R}$$

(c) Sketch the level curve $g(x, y) = 0$.

$$\ln(x^2+y^2) = 0 \Rightarrow x^2+y^2 = 1$$

CIRCLE AT ORIGIN, $r = 1$



(d) Describe the level curve $g(x, y) = 1$.

$$\ln(x^2+y^2) = 1$$

$$x^2+y^2 = e \leftarrow$$ CIRCLE AT ORIGIN, $r = \sqrt{e}$

(e) At which points is g continuous?

g IS CONTINUOUS ON ITS ENTIRE DOMAIN $\{(x,y) : (x,y) \neq (0,0)\}$

(f) Rewrite the equation $z = \ln(x^2 + y^2)$ in cylindrical coordinates.

$$z = \ln(r^2) = 2\ln r$$

OR $e^{z/2} = r$

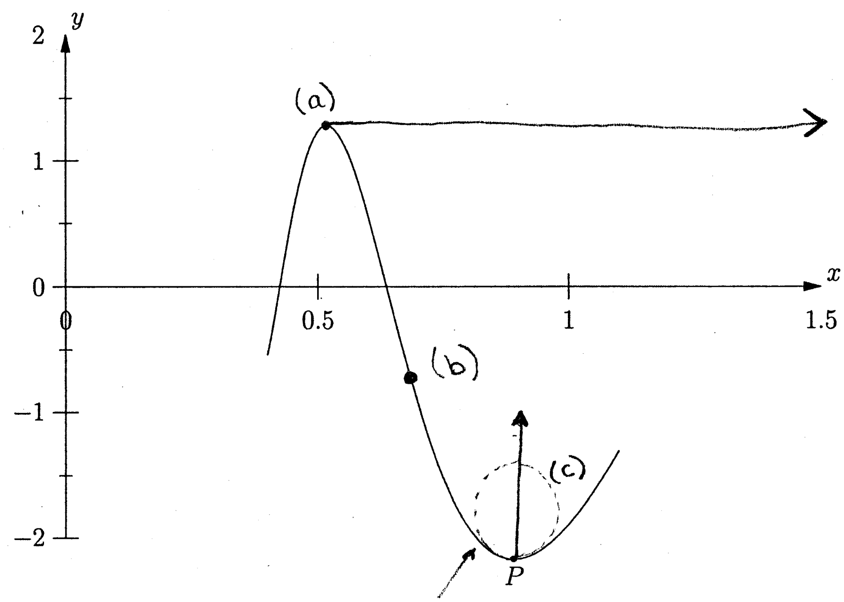
11. (6 points) Suppose a particle moves along the curve from left to right. Sketch and label each of the following:

(a) the unit tangent vector at the point of greatest curvature

(b) a point where the principal unit normal vector does not exist

(c) the principal unit normal vector at the point P

(d) (1 pt ex cred) The circle of curvature at the P



(d) IS DASHED.

SHOULD BE TANGENT TO

THE GRAPH AT P.

Math 173 - Test 2b

March 18, 2010

Name: key

Score

Show all work. Supply explanations when necessary.

1. (8 points) The unit binormal vector for a space curve is the cross product of the unit tangent vector and the principal unit normal vector, $\hat{B} = \hat{T} \times \hat{N}$. Find the unit binormal vector for the space curve described by the vector-valued function

$$\vec{r}(t) = (3\cos t)\hat{\imath} + (3\sin t)\hat{\jmath} + 4t\hat{k}.$$

$$\vec{r}\,'(t) = -3\sin t\,\hat{\imath} + 3\cos t\,\hat{\jmath} + 4\hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$$

$$\hat{T}(t) = -\frac{3}{5}\sin t\,\hat{\imath} + \frac{3}{5}\cos t\,\hat{\jmath} + \frac{4}{5}\hat{k}$$

$$\hat{T}'(t) = -\frac{3}{5}\cos t\,\hat{\imath} - \frac{3}{5}\sin t\,\hat{\jmath}$$

$$|\hat{T}'(t)| = \sqrt{\frac{9}{25}\cos^2 t + \frac{9}{25}\sin^2 t} = \frac{3}{5}$$

$$\hat{N}(t) = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$$

$$\hat{T} \times \hat{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left(\frac{4}{5}\sin t\right)\hat{\imath} - \left(\frac{4}{5}\cos t\right)\hat{\jmath} + \left(\frac{3}{5}\sin^2 t + \frac{3}{5}\cos^2 t\right)\hat{k}$$

$$\boxed{\hat{B}(t) = \frac{4}{5}\sin t\,\hat{\imath} - \frac{4}{5}\cos t\,\hat{\jmath} + \frac{3}{5}\hat{k}}$$

2. (1 point) When computing the **unit** binormal vector, there is no need to normalize your final result. Why not?

IT ALREADY HAS MAGNITUDE 1: $|\hat{T} \times \hat{N}| = |\hat{T}|\,|\hat{N}|\,|\sin\theta| = 1 \cdot 1 \cdot \underbrace{1}_{\text{SINCE } \hat{T} \text{ AND } \hat{N} \text{ ARE ORTHOG.}}$

3. (3 points) For the function $f(x,y,z)$, the gradient vector is the vector of partial derivatives:

$$\text{grad } f(x,y,z) = f_x(x,y,z)\hat{\imath} + f_y(x,y,z)\hat{\jmath} + f_z(x,y,z)\hat{k}.$$

Compute the gradient vector for $f(x,y,z) = \dfrac{3xz}{x+y}$.

$$\text{GRAD } f(x,y,z) = \left(\frac{(x+y)(3z) - (3xz)(1)}{(x+y)^2}\right)\hat{\imath} + \left(-\frac{3xz}{(x+y)^2}\right)\hat{\jmath} + \left(\frac{3x}{x+y}\right)\hat{k}$$

$$\boxed{\text{GRAD } f(x,y,z) = \frac{3yz}{(x+y)^2}\hat{\imath} - \frac{3xz}{(x+y)^2}\hat{\jmath} + \frac{3x}{x+y}\hat{k}}$$

4. (3 points) Convert $\rho \sin^2\phi = \cos\phi$ to an equation in rectangular coordinates. (It may be useful to use $\cos^2\theta + \sin^2\theta = 1$.)

$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

$$\rho \sin^2\varphi = \cos\varphi$$

$$\rho \sin^2\varphi(\cos^2\theta + \sin^2\theta) = \cos\varphi$$

$$\rho^2 \sin\varphi(\cos^2\theta + \sin^2\theta) = \rho\cos\varphi$$

$$\boxed{x^2 + y^2 = z}$$