Math 173 - 1st Final Exam

May 6, 2010

Name key

Score

Show all work. Supply explanations when necessary. Unless otherwise specified, you may use your calculator to evaluate any integrals. Each problem is worth 12 points (unless otherwise indicated).

1. (14 points) Evaluate the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{3x-3y}{y^2-x^2} = \lim_{(x,y)\to(1,1)} \frac{3(x-y)}{(y-x)(y+x)} = \lim_{(x,y)\to(1,1)} \frac{-3}{y+x} = \boxed{-\frac{3}{2}}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x-y^2}{2x^2+y}$

Along $x=0$: $\displaystyle\lim_{y\to 0} \frac{-y^2}{y} = \lim_{y\to 0}(-y) = 0$

Along $y=x$: $\displaystyle\lim_{x\to 0} \frac{2x-x^2}{2x^2+x} = \lim_{x\to 0} \frac{x(2-x)}{x(2x+1)} = \frac{2}{1} = 2$

Limit DNE.

2. Find an equation of the plane containing the points $(2,1,1)$, $(0,4,1)$, and $(-2,1,4)$.

P Q R

$\vec{PQ} = -2\hat{i} + 3\hat{j}$

$\vec{PR} = -4\hat{i} + 3\hat{k}$

$$\vec{N} = \vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix}$$

$= \hat{i}(9) - \hat{j}(-6) + \hat{k}(12)$

$= 9\hat{i} + 6\hat{j} + 12\hat{k}$

Point $(2,1,1)$

$\vec{N} = 9\hat{i} + 6\hat{j} + 12\hat{k}$

$9(x-2) + 6(y-1) + 12(z-1) = 0$

OR

$9x + 6y + 12z = 36$

3. Find and classify all relative extreme values and saddle points.

$$f(x,y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$$

$$f_x(x,y) = 2x - 2y - 2 = 0 \qquad x - y = 1$$
$$\Rightarrow \qquad -x + 2y = -1$$
$$f_y(x,y) = -2x + 4y + 2 = 0$$

$$y = 0$$
$$x = 1$$

$$D(x,y) = \begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 8 - 4 = 4$$

$$D(1,0) = 4 \text{ AND } f_{xx}(1,0) = 2 > 0 \Rightarrow \boxed{f(1,0) = 0 \text{ IS A RELATIVE MIN}}$$

4. Let $\vec{x} = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{y} = -5\hat{\imath} + 2\hat{\jmath} - \hat{k}$.

(a) Find the angle between $\vec{x}$ and $\vec{y}$.

$$\cos\theta = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|} = \frac{-15 + 2 + 2}{\sqrt{14}\sqrt{30}} = \frac{-11}{\sqrt{420}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-11}{\sqrt{420}}\right) \approx 2.137} = 122.46^\circ$$

(b) Find the projection of $\vec{x}$ onto $\vec{y}$.

$$\text{proj}_{\vec{y}}\,\vec{x} = \frac{\vec{x}\cdot\vec{y}}{\vec{y}\cdot\vec{y}}\,\vec{y} = \frac{-11}{30}\,\vec{y}$$

$$= \boxed{\frac{-11}{30}\left(-5\hat{\imath} + 2\hat{\jmath} - \hat{k}\right)}$$

5. The temperature at the point (x, y, z) is given by

$$T(x, y, z) = xz^2 \cos(\pi y)$$

where T is measured in °C and x, y, and z in centimeters.

(a) At the point $(2, -1, 2)$, in what direction does the temperature increase the fastest?

$\vec{\nabla} T(x,y,z) = z^2 \cos \pi y \,\hat{i} - \pi x z^2 \sin \pi y \,\hat{j} + 2xz \cos \pi y \,\hat{k}$

$\vec{\nabla} T(2,-1,2) = 4 \cos(-\pi)\,\hat{i} - 8\pi \sin(-\pi)\,\hat{j} + 8 \cos(-\pi)\,\hat{k}$

$= \boxed{-4\hat{i} - 8\hat{k}}$

(b) Find the rate of change of temperature at the point $(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

$\vec{u} = \hat{i} - 2\hat{j} + \hat{k}$

$|\vec{u}| = \sqrt{6}$

$\frac{1}{|\vec{u}|} \vec{\nabla} T(2,-1,2) \cdot \vec{u}$

$= \frac{1}{\sqrt{6}}(-4 - 8) = \boxed{\frac{-12}{\sqrt{6}}}$

6. (15 points) Let P and Q be the points $(2, 1, 7)$ and $(-2, -3, 4)$, respectively.

(a) Find a vector of length 7 in the direction of $\overrightarrow{PQ}$.

$\overrightarrow{PQ} = -4\hat{i} - 4\hat{j} - 3\hat{k}$

$|\overrightarrow{PQ}| = \sqrt{41}$

$\boxed{-\frac{7}{\sqrt{41}}(4\hat{i} + 4\hat{j} + 3\hat{k})}$

(b) Find a unit vector in the xy-plane that is orthogonal to $\overrightarrow{PQ}$.

WE NEED $\overrightarrow{PQ} \cdot \vec{u} = 0$

So $\vec{u} = \hat{i} - \hat{j}$ AND NORMALIZE $\boxed{\frac{1}{\sqrt{2}}(\hat{i} - \hat{j})}$

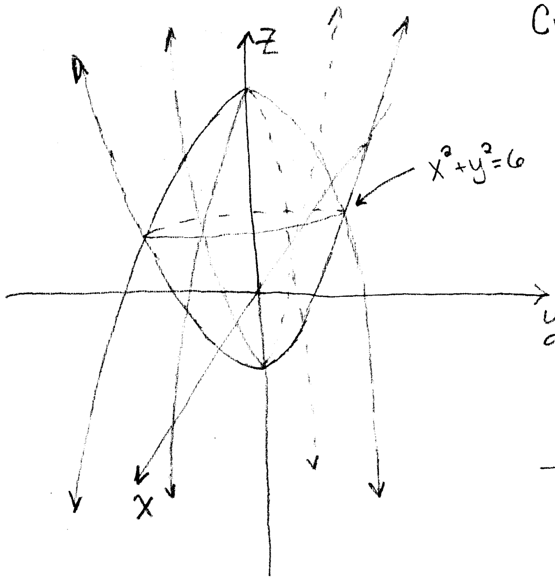
(c) Find a set of parametric equations for the line segment from P to Q.

$x = 2 - 4t$

$y = 1 - 4t$ $\quad 0 \le t \le 1$

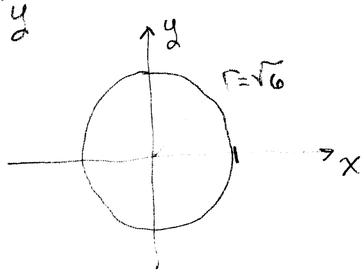
$z = 7 - 3t$

7. A solid lies between the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2 - 4$. The density of the solid at the point (x, y, z) is given by $\rho(x, y, z) = z + 5$. Find the z-coordinate of the center of mass.



CIRCLE OF INTERSECTION IS

$$8 - x^2 - y^2 = x^2 + y^2 - 4$$
$$12 = 2x^2 + 2y^2$$
$$6 = r^2$$
$$r = \sqrt{6}$$



$$M = \int_0^{2\pi} \int_0^{\sqrt{6}} \int_{r^2-4}^{8-r^2} r(z+5)\, dz\, dr\, d\theta = 252\pi$$

$$M_{xy} = \int_0^{2\pi} \int_0^{\sqrt{6}} \int_{r^2-4}^{8-r^2} rz(z+5)\, dz\, dr\, d\theta = 720\pi$$

$$\boxed{\frac{M_{xy}}{M} = \frac{720}{252} = \frac{20}{7}}$$

8. A particle is moving along the space curve described by

$$\vec{r}(t) = 2 \sin t\hat{\imath} + 3 \cos 2t\hat{\jmath} + t^2\hat{k}.$$

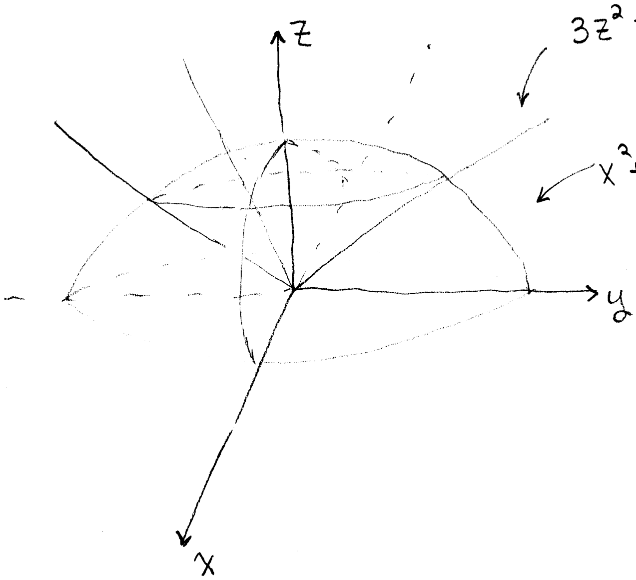
Set up the definite integral that gives the length of the curve from the point where $t = 0$ to the point where $t = 2$. Use your calculator to approximate the value of the integral. (Make sure your calculator is in radian mode.)

$$\vec{v}(t) = 2\cos t\,\hat{\imath} - 6\sin 2t\,\hat{\jmath} + 2t\,\hat{k}$$

$$\text{ARC LENGTH} = \int_0^2 \sqrt{4\cos^2 t + 36\sin^2 2t + 4t^2}\, dt$$

$$\approx \boxed{9.1856}$$

9. (13 points) Use spherical coordinates to find the volume of the space region inside the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$ and below the cone $3z^2 = x^2 + y^2$.



$$\left.\begin{array}{l} 3z^2 = x^2 + y^2 \\ x^2 + y^2 + z^2 = 1 \end{array}\right\}$$

INTERSECTION:

$$4z^2 = 1$$

$$\Rightarrow z = \frac{1}{2} = 1 \cos \varphi$$

$$\Rightarrow \varphi = \cos^{-1} \frac{1}{2}$$

$$\Rightarrow \varphi = \frac{\pi}{3}$$

$$\int_{0}^{2\pi} \int_{\varphi = \frac{\pi}{3}}^{\varphi = \frac{\pi}{2}} \int_{\rho = 0}^{\rho = 1} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$$

$$= \boxed{\frac{\pi}{3}}$$

10. Suppose w depends on x, y, and z according to $w = x\cos(yz)$ and x, y, and z depend on s and t:

$$x = s^2, \qquad y = t^2, \qquad z = s - 2t$$

Find $\partial w/\partial t$ at $(s,t) = (\pi, 0)$.

$X = \pi^2$
$y = 0$
$z = \pi$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$= (\cos yz)(0) + (-xz\sin yz)\overset{0}{(2t)} + \overset{0}{(-xy\sin yz)}(-2)$$

$$= 0 + 0 + 0$$

$$= \boxed{0}$$

11. The quarterback of a football team throws the ball with an initial speed of 54 feet per second, at an angle of 35°, and at a height of 7 feet above the playing field. How far downfield has the ball traveled at the moment when it reaches its maximum height? (Ignore air resistance and use $g = 32\,\mathrm{ft/sec^2}$.)

$$\vec{r}(t) = (54\cos 35°\, t)\,\hat{i} + (-16t^2 + 54\sin 35°\, t + 7)\,\hat{j}$$

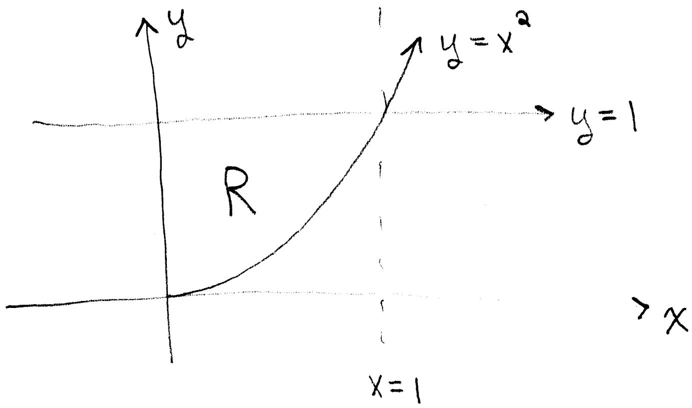
$$-32t + 54\sin 35° = 0$$

$$\Rightarrow t = \frac{54\sin 35°}{32}$$

$$\boxed{(54\cos 35°)\left(\frac{54\cos 35°}{32}\right) = 42.815\ \text{FT}}$$

12. Sketch the region of integration, reverse the order of integration, and **evaluate the iterated integral by hand.**

$$\int_0^1 \int_{x^2}^1 x^3 \sin y^3 \, dy \, dx$$



$$\int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{y}} x^3 \sin y^3 \, dx \, dy$$

$$= \int_0^1 \frac{1}{4} x^4 \sin y^3 \Big|_0^{\sqrt{y}} dy$$

$$= \frac{1}{4} \int_0^1 y^2 \sin y^3 \, dy$$

$u = y^3$

$du = 3y^2 \, dy$

$\frac{1}{3} du = y^2 \, dy$

$$\frac{1}{12} \int_0^1 \sin u \, du$$

$$= \frac{1}{12} (\cos 0 - \cos 1)$$

$$= \boxed{\frac{1}{12} (1 - \cos 1)}$$