

Math 173 - 2nd Final Exam
 May 18, 2010

Name key
 Score _____

Show all work. Supply explanations when necessary. Unless otherwise specified, you may use your calculator to evaluate any integrals. Each problem is worth 10 points.

1. Sketch the region of integration, reverse the order of integration, and evaluate the iterated integral by hand.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} dy dx = \int_0^2 \int_0^{y^2} \frac{x}{y^5+1} dx dy$$

$$= \frac{1}{2} \int_0^2 \frac{y^4}{y^5+1} dy$$

$$= \frac{1}{10} \int_1^{33} \frac{1}{u} du = \boxed{\frac{\ln 33}{10}}$$

$u = y^5 + 1$
 $du = 5y^4 dy$
 $\frac{1}{5} du = y^4 dy$

2. Find and classify all relative extreme values of the function $f(x, y)$.

$$f(x, y) = x^2 + xy + 2y^2 + x - 3y + 10$$

$$f_x(x, y) = 2x + y + 1 = 0$$

$$f_y(x, y) = x + 4y - 3 = 0$$

Top x 4 AND SUBTRACT

$$\Rightarrow 7x + 7 = 0$$

$$x = -1$$

$$2(-1) + y + 1 = 0$$

$$\Rightarrow y = 1$$

Only one CRIT PT:

$$(-1, 1)$$

$$D(x, y) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7$$

$$D(-1, 1) = 7 > 0 \text{ AND } f_{xx}(-1, 1) = 2 > 0$$

$$\Rightarrow \boxed{f(-1, 1) = 8}$$

IS A RELATIVE MIN

3. A plane passes through the points $P(2, 1, 3)$, $Q(-7, 6, -1)$ and $R(3, 0, -1)$. Find a set of parametric equations for the line normal to the plane and passing through $(2, 1, 3)$.

$$\vec{PQ} = -9\hat{i} + 5\hat{j} - 4\hat{k}$$

$$\vec{PR} = \hat{i} - \hat{j} - 4\hat{k}$$

$$\begin{aligned} \vec{N} = \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -9 & 5 & -4 \\ 1 & -1 & -4 \end{vmatrix} \\ &= \hat{i}(-24) - \hat{j}(40) + \hat{k}(4) \\ &= -24\hat{i} - 40\hat{j} + 4\hat{k} \end{aligned}$$

Using $P(2, 1, 3)$:

$$\begin{aligned} x &= -24t + 2 \\ y &= -40t + 1 \\ z &= 4t + 3 \end{aligned}$$

4. Find a vector of length 3 that is normal to the plane given by $5x - 3y + 2z = 10$.

$$\vec{N} = 5\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\begin{aligned} |\vec{N}| &= \sqrt{25 + 9 + 4} \\ &= \sqrt{38} \end{aligned}$$

$$\frac{3}{\sqrt{38}} (5\hat{i} - 3\hat{j} + 2\hat{k})$$

5. Let $\vec{r}(t) = t^2\hat{i} + (\sin t - t \cos t)\hat{j} + (\cos t + t \sin t)\hat{k}$. Find the unit tangent vector and the principal unit normal vector. (Hint: If you're doing everything correctly, the computations should not be messy.)

$$\vec{r}'(t) = 2t\hat{i} + t \sin t \hat{j} + t \cos t \hat{k}$$

$$|\vec{r}'(t)| = \sqrt{4t^2 + t^2} = \sqrt{5}t \text{ Assuming } t > 0$$

$$\hat{T}(t) = \frac{1}{\sqrt{5}} (2\hat{i} + \sin t \hat{j} + \cos t \hat{k})$$

$$\hat{T}'(t) = \frac{1}{\sqrt{5}} (\cos t \hat{j} - \sin t \hat{k})$$

$$|\hat{T}'(t)| = \frac{1}{\sqrt{5}}$$

$$\hat{N}(t) = \cos t \hat{j} - \sin t \hat{k}$$

6. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the line segment from $(0, 1, 3)$ to $(2, -3, 4)$ and $\vec{F}(x, y, z) = y\hat{i} + (x^2 + 1)\hat{j} + z^2\hat{k}$.

$$\vec{v} = 2\hat{i} - 4\hat{j} + \hat{k}$$

$$C: x = 2t$$

$$y = -4t + 1$$

$$z = t + 3$$

$$0 \leq t \leq 1$$

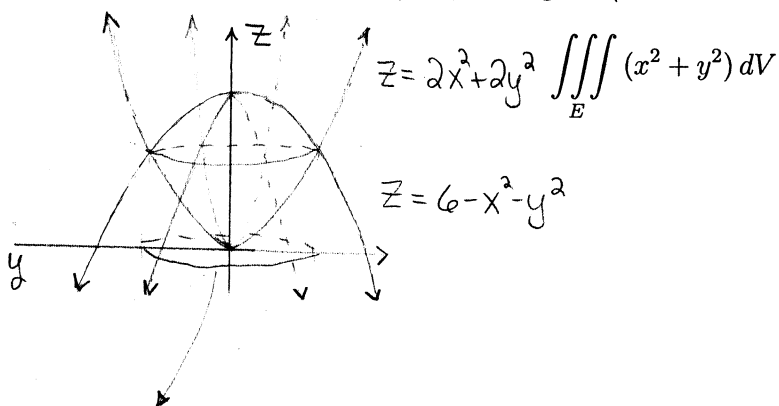
$$\int_C y dx + (x^2 + 1) dy + z^2 dz$$

$$= \int_0^1 (-4t + 1)(2 dt) + (4t^2 + 1)(-4 dt) + (t + 3)^2 dt$$

$$= \int_0^1 (-8t + 2 - 16t^2 - 4 + t^2 + 6t + 9) dt = \int_0^1 (-15t^2 - 2t + 7) dt$$

$$= -5 - 1 + 7 = \boxed{1}$$

7. Let E be the space region bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = 6 - x^2 - y^2$. Evaluate the following triple integral. (Hint: It's easier in cylindrical coordinates.)



$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{2r^2}^{6-r^2} r^3 dz dr d\theta$$

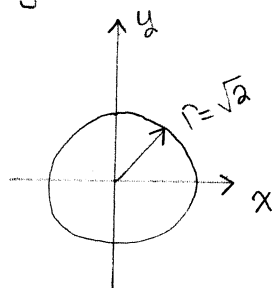
$$= 4\pi$$

$$2x^2 + 2y^2 = 6 - x^2 - y^2$$

$$3r^2 = 6$$

$$r^2 = 2$$

$$dA = r dr d\theta$$



8. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with a speed of 115 ft/sec at an angle of 50° above the horizontal. Does the ball clear the fence? (Use $g = 32 \text{ ft/sec}^2$.)

$$\vec{r}(t) = (115 \cos 50^\circ t) \hat{i} + (-16t^2 + 115 \sin 50^\circ t + 3) \hat{j}$$

$$400 = 115 \cos 50^\circ t$$

$$t = \frac{400}{115 \cos 50^\circ} \approx 5.41 \text{ sec}$$

THIS t-value

$$-16t^2 + 115 \sin 50^\circ t + 3$$

$$= 11.20 \text{ FT}$$

4

YES, IT CLEARS THE
FENCE BY 1.20 FT

9. Use Lagrange multipliers to find the extreme values of $f(x, y) = \frac{1}{3}x^3 + y^2$ on the unit circle $x^2 + y^2 = 1$.

$$f(x, y) = \frac{1}{3}x^3 + y^2$$

$$g(x, y) = x^2 + y^2$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$x^2 + y^2 = 1$$

$$x^2 = 2\lambda x$$

$$2y = 2\lambda y \Rightarrow 2y(1-\lambda) = 0$$

$$x^2 + y^2 = 1$$

$$y = 0 \text{ or } \lambda = 1$$

$$x = \pm 1$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0 \quad x = 2$$

NOT POSSIBLE

$$y = \pm 1$$

CRIT PTS ARE

$$(1, 0), (-1, 0),$$

$$(0, 1), (0, -1)$$

$$M_{\min} = f(-1, 0) = -\frac{1}{3}$$

$$M_{\max} = f(0, \pm 1) = 1$$

10. The curves described by the vector-valued functions $\vec{r}_1(t) = t\hat{i} + (t^2 + 3)\hat{j} + (6 - 5t)\hat{k}$ and $\vec{r}_2(t) = t^2\hat{i} + (2t + 2)\hat{j} + t^3\hat{k}$ intersect at the point $(1, 4, 1)$. Find the angle of intersection of the curves. (Hint: Find the angle between the velocity vectors at the point of intersection.)

$$\vec{v}_1(t) = \hat{i} + 2t\hat{j} - 5\hat{k}$$

$$\vec{v}_1(1) = \hat{i} + 2\hat{j} - 5\hat{k}$$

$$\vec{v}_2(t) = 2t\hat{i} + 2\hat{j} + 3t^2\hat{k}$$

$$\vec{v}_2(1) = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\cos \theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| |\vec{v}_2|} = \frac{2 + 4 - 15}{\sqrt{30} \sqrt{17}}$$

$$\cos \theta = \frac{-9}{\sqrt{510}} \Rightarrow \theta \approx 113.5^\circ$$

11. The temperature at a point (x, y) is given by

$$T(x, y) = \frac{xy}{x^2 + y^2 + 1}$$

where T is measured in $^{\circ}\text{C}$ and x and y in meters.

(a) At the point $(1, 1)$, in what direction does the temperature decrease the fastest?

$$\vec{\nabla} T(x, y) = \frac{(x^2 + y^2 + 1)(y) - (xy)(2x)}{(x^2 + y^2 + 1)^2} \hat{i} + \frac{(x^2 + y^2 + 1)(x) - (xy)(2y)}{(x^2 + y^2 + 1)^2} \hat{j}$$

$$\vec{\nabla} T(1, 1) = \frac{1}{9} \hat{i} + \frac{1}{9} \hat{j}$$

$$-\vec{\nabla} T(1, 1) = -\frac{1}{9} \hat{i} - \frac{1}{9} \hat{j}$$

OPPOSITE
 $\vec{\nabla} T(1, 1)$

(b) Find the rate of change of temperature at the point $(1, 1)$ in the direction toward the point $(2, 3)$.

$$\vec{u} = \hat{i} + 2\hat{j}, \quad |\vec{u}| = \sqrt{5}$$

$$D_{\vec{u}} T(1, 1) = \frac{1}{\sqrt{5}} \left(\frac{1}{9} + \frac{2}{9} \right) = \frac{3}{9\sqrt{5}} = \frac{1}{3\sqrt{5}}$$

12. Show that the limit does not exist. (Hint: $y = x^2$ is a path through $(0, 0)$.)

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y}{x^4 + y^2}$$

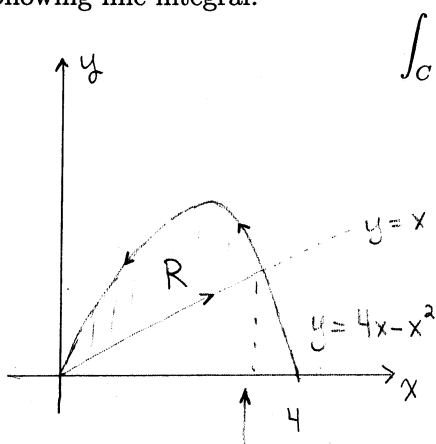
Along $y = 0$: $\lim_{x \rightarrow 0} \frac{0}{x^4} = 0$

Along $y = x^2$: $\lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \frac{1}{2}$

Limit DNE

By Two-PATH
TEST

13. Let R be the plane region bounded by the graphs of $y = x$ and $y = 4x - x^2$. Let C be the region's positively-oriented boundary curve. Use Green's theorem to evaluate the following line integral.



$$\int_C xy dx + x^2 y^3 dy$$

$$\int_0^3 \int_x^{4x-x^2} (2xy^3 - x) dy dx$$

$$= \frac{44253}{140}$$

$$4x - x^2 = x$$

$$x^2 - 3x = 0$$

$$x(x-3) = 0$$

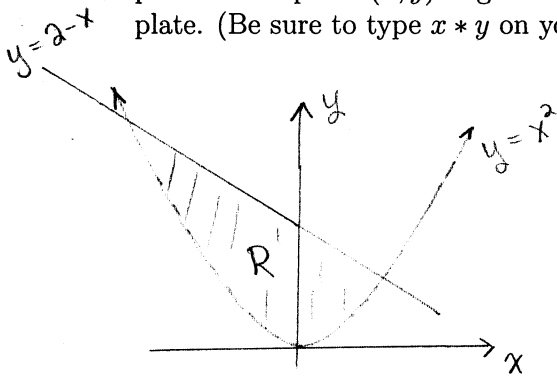
$$x=0, x=3$$

14. Suppose u is a function of x , y , and z , where x , y , and z are functions of r , s , and t . State the chain rule formulas for $\partial u / \partial r$ and $\partial u / \partial t$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}$$

15. A thin plate is bounded by the graphs of $y = x^2$ and $y = 2 - x$. The density of the plate at the point (x, y) is given by $\rho(x, y) = 5 + xy$. Find the center of mass of the plate. (Be sure to type $x * y$ on your calculator.)



$$x^2 = 2 - x$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x = -2, x = 1$$

$$M = \int_{-2}^1 \int_{x^2}^{2-x} (5+xy) dy dx = \frac{135}{8}$$

$$M_y = \int_{-2}^1 \int_{x^2}^{2-x} x(5+xy) dy dx = \frac{-513}{140}$$

$$M_x = \int_{-2}^1 \int_{x^2}^{2-x} y(5+xy) dy dx = \frac{837}{40}$$

$$CM = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) \approx \boxed{(-0.217, 1.24)}$$