

Math 173 - Test 2
March 17, 2011

Name key _____
Score _____

Show all work. Supply explanations when necessary.

1. (10 points) Suppose $z = 2xe^{5y} - 3ye^{-x}$.

- (a) Which first partial derivative should be computed first in order to obtain $\frac{\partial^2 z}{\partial x \partial y}$?

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \quad \begin{matrix} \text{THE } y\text{-PARTIAL} \\ \text{COMES FIRST.} \end{matrix}$$

- (b) Do you expect to have $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$? Explain.

Yes. THE MIXED PARTIALS WILL BE EQUAL

IF THEY ARE CONTINUOUS. FOR THIS FUNCTION, PARTIALS OF

- (c) Compute $\frac{\partial^2 z}{\partial x \partial y}$.

ALL ORDERS WILL BE CONTINUOUS.

$$\frac{\partial z}{\partial y} = 10xe^{5y} - 3e^{-x}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \boxed{10e^{5y} + 3e^{-x}}$$

2. (4 points) In a sentence or so, explain what is happening to an accelerating object whose tangential component of acceleration is zero.

THE OBJECT'S SPEED MUST BE CONSTANT.

ONLY THE DIRECTION OF MOTION IS CHANGING.

3. (10 points) Find the principal unit normal vector at $t = \frac{3\pi}{4}$.

$$\vec{r}(t) = 6 \cos t \hat{i} + 2 \hat{j} + 6 \sin t \hat{k}$$

$$\vec{r}'(t) = -6 \sin t \hat{i} + 6 \cos t \hat{k}$$

$$|\vec{r}'(t)| = \sqrt{36 \sin^2 t + 36 \cos^2 t} = 6$$

$$\hat{T}(t) = -\sin t \hat{i} + \cos t \hat{k}$$

$$\hat{T}'(t) = -\cos t \hat{i} - \sin t \hat{k}$$

↑ Already A UNIT VECTOR

$$\hat{N}(t) = -\cos t \hat{i} - \sin t \hat{k}$$

$$\boxed{\hat{N}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \hat{i} - \frac{\sqrt{2}}{2} \hat{k}}$$

4. (8 points) Let $w = f(x, y, z) = x^2yz^2 + xy \cos z$ and $d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$.

- (a) Compute $\vec{\nabla}f(x, y, z) \cdot d\vec{r}$.

$$\begin{aligned} \vec{\nabla}f(x, y, z) &= (2xyz^2 + y \cos z) \hat{i} + (x^2z^2 + x \cos z) \hat{j} \\ &\quad + (2x^2yz - xy \sin z) \hat{k} \end{aligned}$$

$$\vec{\nabla}f(x, y, z) \cdot d\vec{r} =$$

$$\boxed{(2xyz^2 + y \cos z) dx + (x^2z^2 + x \cos z) dy + (2x^2yz - xy \sin z) dz}$$

- (b) Is your answer in part (a) the same as the total differential dw ?

$$\underline{\text{YES}}, \quad dw = f_x dx + f_y dy + f_z dz$$

EXACTLY THE DOT PRODUCT

5. (5 points) Suppose w is a function of x, y, z and x, y, z are functions of u, v . Write the chain rule formula $\frac{\partial w}{\partial u}$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

6. (10 points) Assume that $z - e^x \sin(2y + 3z) = 0$ implicitly defines z as a function of x and y . Find both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$F(x, y, z) = z - e^x \sin(2y + 3z)$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \boxed{\frac{e^x(\sin 2y + 3z)}{1 - 3e^x \cos(2y + 3z)}}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \boxed{\frac{2e^x \cos(2y + 3z)}{1 - 3e^x \cos(2y + 3z)}}$$

7. (10 points) Find an equation of the plane tangent to the surface $y \ln(xz^2) = 2$ at the point $(e, 2, 1)$.

$$\text{Let } F(x, y, z) = y \ln(xz^2) = y \ln x + 2y \ln z$$

$y \ln(xz^2) = 2$ is the level curve $F(x, y, z) = 2$ passing through $(e, 2, 1)$.

\Rightarrow Gradient gives normal vector.

$$\vec{\nabla} F(x, y, z) = \frac{y}{x} \hat{i} + (\ln x + 2\ln z) \hat{j} + \frac{2y}{z} \hat{k}$$

$$\vec{N} = \vec{\nabla} F(e, 2, 1) = \frac{2}{e} \hat{i} + \hat{j} + 4 \hat{k}$$

Point $(e, 2, 1)$

\Rightarrow

Plane is

$$\frac{2}{e}(x-e) + (y-2) + 4(z-1) = 0$$

8. (8 points) Find the length of the space curve described by

$$\vec{r}(t) = -\sin 3t \hat{i} + \cos 3t \hat{j} + 5t \hat{k}$$

over the interval $[0, \pi/3]$.

$$\vec{r}'(t) = -3 \cos 3t \hat{i} - 3 \sin 3t \hat{j} + 5 \hat{k}$$

$$|\vec{r}'(t)| = \sqrt{9 \cos^2 3t + 9 \sin^2 3t + 25} = \sqrt{34}$$

$$\text{Length} = \int_0^{\pi/3} \sqrt{34} dt = \boxed{\frac{\sqrt{34} \pi}{3}}$$

9. (10 points) Find each limit or show that the limit does not exist.

$$(a) \lim_{(x,y) \rightarrow (2,2)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y) \rightarrow (2,2)} \frac{(x+y)(x-y)}{(x-y)} = \boxed{4}$$

$$(b) \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$$

Along $y=0$: $\lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = 0$

Along $y=x$: $\lim_{x \rightarrow 0} \frac{x^2}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$

} DIFFERENT
LIMITS ALONG
DIFFERENT PATHS.

Limit DNE

10. (10 points) Find the directional derivative of $h(x, y, z) = \ln(x + y + z)$ at $(1, 2, 1)$ in the direction of $\vec{v} = 2\hat{i} + 3\hat{j} + \hat{k}$.

$$\vec{\nabla} h(x, y, z) = \frac{\hat{i}}{x+y+z} + \frac{\hat{j}}{x+y+z} + \frac{\hat{k}}{x+y+z}$$

$$\vec{\nabla} h(1, 2, 1) = \frac{1}{4} (\hat{i} + \hat{j} + \hat{k})$$

$$D_{\vec{v}} h(1, 2, 1) = \frac{1}{|\vec{v}|} (\vec{\nabla} h \cdot \vec{v})$$

$$|\vec{v}| = \sqrt{4+9+1} = \sqrt{14}$$

$$\vec{v} = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\frac{1}{\sqrt{14}} \left(\frac{2}{4} + \frac{3}{4} + \frac{1}{4} \right) = \boxed{\frac{3}{2\sqrt{14}}}$$

Follow-up question: At the point $(1, 2, 1)$, in what direction does h increase most rapidly?

| IN THE DIRECTION OF THE GRADIENT VECTOR

i.e. $\hat{i} + \hat{j} + \hat{k}$.

11. (10 points) Consider the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.

(a) What is the domain of f ?

$$4 - x^2 - y^2 \geq 0$$

$$\Rightarrow \text{Domain} = \{(x, y) : x^2 + y^2 \leq 4\}$$

(b) What is the range of f ?

$$\{z : 0 \leq z \leq 2\}$$

(c) Is the domain open, closed, both, or neither?

CLOSED, IT CONTAINS ITS
BOUNDARY.

(d) Sketch or describe (in detail) the level curve $f(x, y) = 0$.

$$f(x, y) = 0 \Rightarrow 4 - x^2 - y^2 = 0$$

$$\Rightarrow x^2 + y^2 = 4$$

CIRCLE WITH
RADIUS 2
CENTERED AT
(0,0).

(e) Sketch or describe (in detail) the level curve $f(x, y) = \sqrt{3}$.

$$f(x, y) = \sqrt{3} \Rightarrow$$

$$3 = 4 - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 = 1$$

CIRCLE WITH
RADIUS 1 CENTERED
AT (0,0)

12. (5 points) Suppose a particle moves along the curve from left to right. Label each of the following:

- (a) a point at which the curvature is a maximum
- (b) a point where the principal unit normal vector does not exist
- (c) a point at which the curvature is a minimum
- (d) a point at which the unit tangent vector is horizontal
- (e) a point at which the principal unit normal vector is vertical (Does it point up or down at your point?)

