

**Math 173 - 2nd Final Exam**  
May 16, 2011

Name key  
Score \_\_\_\_\_

Show all work. Supply explanations when necessary. Unless otherwise specified, you may use your calculator to evaluate any integrals. Each problem is worth 10 points.

1. Find  $\vec{r}(t)$  given that  $\frac{d\vec{r}}{dt} = \frac{1}{1+t^2}\hat{i} + \frac{1}{t^2}\hat{j} + \frac{1}{t}\hat{k}$  and  $\vec{r}(1) = 2\hat{i}$ .

$$\vec{r}(t) = (\tan^{-1} t + c_1)\hat{i} + \left(-\frac{1}{t} + c_2\right)\hat{j} + (\ln|t| + c_3)\hat{k}$$

$$\vec{r}(1) = 2\hat{i} = (\tan^{-1} 1 + c_1)\hat{i} + (-1 + c_2)\hat{j} + (c_3)\hat{k}$$

$$\Rightarrow 2 = \frac{\pi}{4} + c_1$$

$$0 = -1 + c_2$$

$$0 = c_3$$

$$\boxed{\vec{r}(t) = \left(2 - \frac{\pi}{4} + \tan^{-1} t\right)\hat{i} + \left(1 - \frac{1}{t}\right)\hat{j} + \ln|t|\hat{k}}$$

2. Find a vector of magnitude 5 that is orthogonal to both  $\vec{x} = 3\hat{i} - 2\hat{j} + \hat{k}$  and  $\vec{y} = -\hat{i} + 5\hat{j} - 3\hat{k}$ .

$$\begin{aligned} \vec{x} \times \vec{y} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ -1 & 5 & -3 \end{vmatrix} = \hat{i}(1) - \hat{j}(-8) + \hat{k}(13) \\ &= \hat{i} + 8\hat{j} + 13\hat{k} \end{aligned}$$

$$\begin{aligned} |\vec{x} \times \vec{y}| &= \sqrt{1 + 64 + 169} \\ &= \sqrt{234} \end{aligned}$$

$$\boxed{\frac{5}{\sqrt{234}}(\hat{i} + 8\hat{j} + 13\hat{k})}$$

3. Find the limit or show that it does not exist.

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y}{x + y}$$

$$\text{Along } y=0 : \lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Two DIFFERENT }$$

$$\text{Along } x=0 : \lim_{y \rightarrow 0} \frac{y}{y} = \lim_{y \rightarrow 0} 1 = 1 \quad \Rightarrow \underline{\text{Limit ONE}}$$

$$(b) \lim_{(x,y) \rightarrow (2,2)} \frac{x-y}{x^2 - y^2} = \lim_{(x,y) \rightarrow (2,2)} \frac{x-y}{(x+y)(x-y)} =$$

$$\lim_{(x,y) \rightarrow (2,2)} \frac{1}{x+y} = \boxed{\frac{1}{4}}$$

4. Find a set of parametric equations for the line tangent to the graph of  $\vec{r}(t)$  at the point  $(e, 0, 2)$ .

$$\vec{r}(t) = te^t \hat{i} + \sin(\pi t) \hat{j} + \sqrt{3+t^2} \hat{k}$$

$$\vec{r}'(t) = (e^t + te^t) \hat{i} + \pi \cos \pi t \hat{j} + \frac{1}{2} (3+t^2)^{-\frac{1}{2}} (2t) \hat{k}$$

$$\vec{r}'(1) = 2e \hat{i} - \pi \hat{j} + \frac{1}{2} \hat{k}$$

$x = 2et + e$
$y = -\pi t$
$z = \frac{1}{2}t + 2$

5. Find the directional derivative of  $g(x, y, z) = xye^z$  at  $(2, 4, 0)$  in the direction of  $(0, 0, 0)$ .

$$\vec{\nabla}g(x, y, z) = ye^z \hat{i} + xe^z \hat{j} + xy e^z \hat{k}$$

$$\vec{\nabla}g(2, 4, 0) = 4\hat{i} + 2\hat{j} + 8\hat{k}$$

$$\vec{u} = \text{DIRECTION} = (0-2)\hat{i} + (0-4)\hat{j} + (0-0)\hat{k} = -2\hat{i} - 4\hat{j}$$

$$|\vec{u}| = \sqrt{4+16}$$

$$\frac{1}{|\vec{u}|} \vec{\nabla}g \cdot \vec{u} = \frac{1}{\sqrt{20}} (-8 - 8)$$

$$= \frac{-16}{\sqrt{20}} = -\frac{8}{\sqrt{5}}$$

Follow-up: At the point  $(2, 4, 0)$ , in what direction is  $g$  increasing most rapidly?

$$\begin{aligned} &\text{IN THE DIRECTION OF } \vec{\nabla}g(2, 4, 0) \\ &= 4\hat{i} + 2\hat{j} + 8\hat{k}. \end{aligned}$$

6. Use the chain rule to find  $\frac{\partial w}{\partial s}$  when  $s = 4$  and  $t = \pi/4$ .

$$\text{When } s=4, t=\frac{\pi}{4}$$

$$w = 5x^3 - xy^2; \quad x = s \cos t, \quad y = s \sin t$$

$$x = 2\sqrt{2} = y$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} = (15x^2 - y^2) \cos t + (-2xy)(\sin t)$$

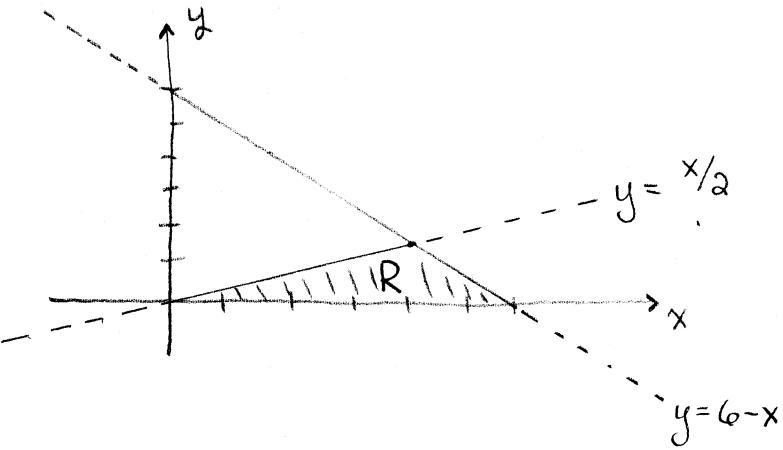
$$\text{At } s=4, t=\frac{\pi}{4},$$

$$\frac{\partial w}{\partial s} = \left[ 15(2\sqrt{2})^2 - (2\sqrt{2})^2 \right] \cdot \frac{\sqrt{2}}{2} + (-2)(2\sqrt{2}) \cdot \frac{\sqrt{2}}{2}$$

$$= 56\sqrt{2} - 8\sqrt{2} = 48\sqrt{2} \approx 67.88$$

7. Sketch the region  $R$  whose area is given by the iterated integral. Then reverse the order of integration and evaluate the new iterated integral by hand.

$$\int_0^4 \int_0^{x/2} dy dx + \int_4^6 \int_0^{6-x} dy dx$$



$$\begin{aligned}
 & \int_{y=0}^{y=2} \int_{x=6-y}^{x=2} dx dy \\
 & = \int_0^2 (6 - 3y) dy \\
 & = \left[ 6y - \frac{3}{2}y^2 \right]_0^2 \\
 & = 12 - 6 = \boxed{6}
 \end{aligned}$$

8. Find and classify all critical points of the function  $f(x, y)$ .

$$f(x, y) = -x^2 - 5y^2 + 10x - 10y - 28$$

$$f_x(x, y) = -2x + 10 = 0 \Rightarrow x = 5 \Rightarrow (5, -1)$$

$$f_y(x, y) = -10y - 10 = 0 \Rightarrow y = -1$$

$$D = \begin{vmatrix} -2 & 0 \\ 0 & -10 \end{vmatrix} = 20$$

$$D(5, -1) = 20 > 0 \text{ AND } f_{xx}(5, -1) = -2 < 0$$

$\Rightarrow \boxed{f(5, -1) = 2 \text{ IS A REL MAX}}$

9. Find the angle between the planes.

$$\begin{aligned}x - 3y + 6z &= 4 \\5x + y - z &= 4\end{aligned}$$

$$\vec{u} = \hat{i} - 3\hat{j} + 6\hat{k}$$

$$\vec{v} = 5\hat{i} + \hat{j} - \hat{k}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{5 - 3 - 6}{\sqrt{46} \sqrt{27}} = \frac{-4}{\sqrt{1242}}$$

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{1242}}\right) \approx 96.52^\circ$$

$$\text{or } 180^\circ - 96.52^\circ = 83.48^\circ$$

10. Evaluate the line integral  $\int_C 2xyz \, ds$ , where  $C$  is the line segment from  $(0, 0, 0)$  to  $(12, 5, 84)$ .

$$\begin{aligned}\vec{u} = 12\hat{i} + 5\hat{j} + 84\hat{k} \Rightarrow x &= 12t \\y &= 5t \quad 0 \leq t \leq 1 \\z &= 84t\end{aligned}$$

$$\begin{aligned}\vec{r}(t) &= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \\|\vec{r}'(t)| &= \sqrt{12^2 + 5^2 + 84^2} = 85\end{aligned}$$

$$\int_C 2xyz \, ds = \int_0^1 10080t^3 \cdot 85 \, dt$$

$$= 214200t^4 \Big|_0^1 = \boxed{214200}$$

11. Use a triple integral in spherical coordinates to find the volume of the upper hemisphere  
 $z = \sqrt{9 - x^2 - y^2}$ .

↑  
 Upper HALF OF  $x^2 + y^2 + z^2 = 3^2$

$$\begin{aligned} \text{Volume} &= \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\rho=0}^{\rho=3} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta \\ &= 2\pi \int_0^{\frac{\pi}{2}} 9 \sin \varphi \, d\varphi = -18\pi \cos \varphi \Big|_0^{\frac{\pi}{2}} \\ &= \boxed{18\pi} \end{aligned}$$

12. Find the unit tangent vector at the point where  $t = \pi/2$ .

$$\vec{r}(t) = 2 \sin t \hat{i} + 2 \cos t \hat{j} + 4 \sin^2 t \hat{k}$$

$$\vec{r}'(t) = 2 \cos t \hat{i} - 2 \sin t \hat{j} + 8 \sin t \cos t \hat{k}$$

$$\vec{r}'\left(\frac{\pi}{2}\right) = -2\hat{j}$$

$$\hat{T}\left(\frac{\pi}{2}\right) = \frac{\vec{r}'\left(\frac{\pi}{2}\right)}{\|\vec{r}'\left(\frac{\pi}{2}\right)\|} = \boxed{-\hat{j}}$$

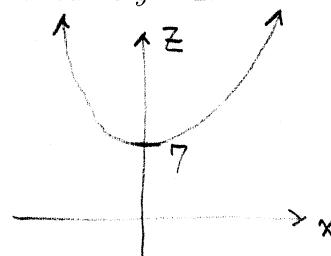
13. Consider the surface described by the equation  $z = x^2 + y^2 + 3$ .

(a) Sketch or describe (in detail) the level curve  $z = 4$ .

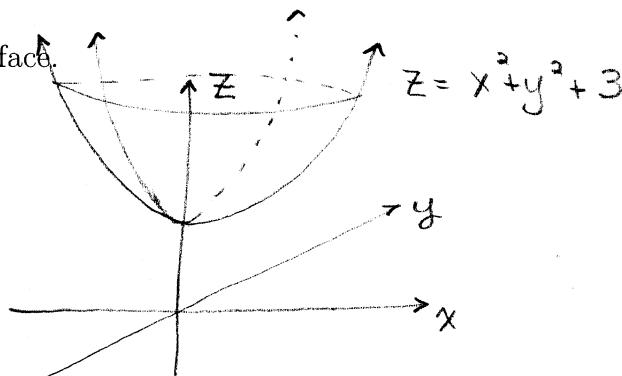
$$z = 4 \Rightarrow x^2 + y^2 = 1 \quad \text{THE UNIT CIRCLE, CENTERED AT } (0,0).$$

(b) Sketch or describe (in detail) the level curve  $y = 2$ .

$$y = 2 \Rightarrow z = x^2 + 7$$



(c) Sketch the surface.



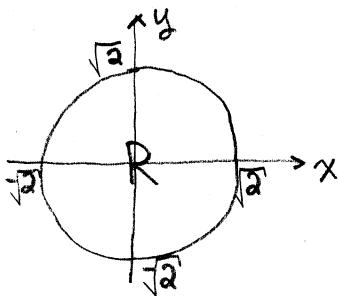
PARABOLOID

OPENING UP WITH  
VERTEX AT  $(0,0,3)$ .

14. A solid lies inside the cylinder  $x^2 + y^2 = 2$  and is bounded by the surfaces  $z = 0$  and  $z = x^2 + y^2 + 3$ . The density of the solid at the point  $(x, y, z)$  is given by  $\rho(x, y, z) = y + z^2 + 1$ . Find the mass of the solid. Use your calculator to evaluate the integral.

REFERRING TO THE FIGURE ABOVE,

$$\begin{aligned} \text{MASS} &= \iiint_R^{\sqrt{x^2+y^2+3}} (y + z^2 + 1) dz dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{r^2+3} (r \sin \theta + z^2 + 1) r dz dr d\theta \\ &= \boxed{\frac{160\pi}{3}} \end{aligned}$$



15. Let  $\vec{F}$  be the conservative vector field  $\vec{F}(x, y, z) = 2xy\hat{i} + (x^2 + z^2)\hat{j} + 2yz\hat{k}$ . Find the corresponding scalar potential function and use it to evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $C$  is a piecewise smooth curve from  $(1, 1, 0)$  to  $(0, 2, 3)$ .

$$\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z)$$

$$\frac{\partial f}{\partial y} = x^2 + z^2 \Rightarrow f(x, y, z) = x^2y + yz^2 + h(x, z)$$

$$\frac{\partial f}{\partial z} = 2yz \Rightarrow f(x, y, z) = yz^2 + c(x, y)$$

IT FOLLOWS THAT

$$f(x, y, z) = x^2y + yz^2 + C$$

$$\int_C \vec{F} \cdot d\vec{r} = f(0, 2, 3) - f(1, 1, 0)$$

$$= 18 - 1 = \boxed{17}$$