Math 173 - Test 1

February 16, 2012

Name <u>key</u>

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) The vector $\vec{u}$ is the 2D vector that has magnitude 7 and makes a 60° angle with the positive x-axis. The vector $\vec{v}$ is the 2D vector that has magnitude 4 and makes an angle of 30° with the positive x-axis. Compute $\vec{u}+\vec{v}$. Then sketch $\vec{u}$, $\vec{v}$, and $\vec{u}+\vec{v}$ in the xy-plane and indicate how these vectors are related by the parallelogram law.

$\vec{u} = 7\cos 60°\hat{i} + 7\sin 60°\hat{j}$

$= 7/2\,\hat{i} + \frac{7\sqrt{3}}{2}\hat{j}$

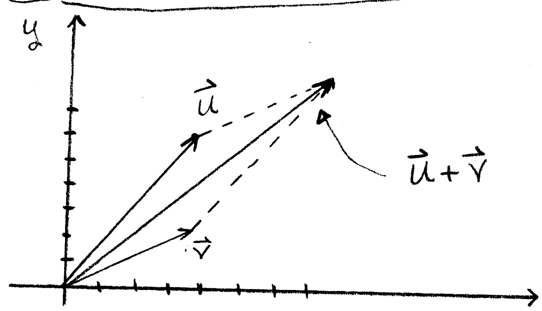
$\approx 3.5\hat{i} + 6.06\hat{j}$

$\vec{v} = 4\cos 30°\hat{i} + 4\sin 30°\hat{j}$

$= 2\sqrt{3}\,\hat{i} + 2\hat{j}$

$\approx 3.46\hat{i} + 2\hat{j}$

$\boxed{\vec{u}+\vec{v} \approx 6.96\hat{i} + 8.06\hat{j}}$



2. (8 points) Given the points P, Q, and R, find the angle between the vectors $\vec{PQ}$ and $\vec{PR}$.

$$P(1,0,-1) \qquad Q(5,2,-10) \qquad R(-1,-8,6)$$

$\vec{PQ} = 4\hat{i} + 2\hat{j} - 9\hat{k}$

$\vec{PR} = -2\hat{i} - 8\hat{j} + 7\hat{k}$

$$\cos\theta = \frac{-8-16-63}{\sqrt{101}\sqrt{117}} = \frac{-87}{\sqrt{11817}} \approx -0.8003$$

$\boxed{\theta \approx 143.16°}$

3. (8 points) Find an equation of the plane passing through the three points in the problem above. Write your final answer in the standard form $ax+by+cz=d$.

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & -9 \\ -2 & -8 & 7 \end{vmatrix}$$

$= \hat{i}(14-72) - \hat{j}(28-18) + \hat{k}(-32+4)$

$= -58\hat{i} - 10\hat{j} - 28\hat{k}$

I'll use $\vec{N} = 29\hat{i} + 5\hat{j} + 14\hat{k}$

AND $P(1,0,-1)$

$29x + 5y + 14z = d$

$29(1) + 5(0) + 14(-1) = 15$

$\boxed{29x + 5y + 14z = 15}$

4. (8 points) Find the principal unit normal vector at $t = 1$ for the following vector-valued function:

$$\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath}$$

$\vec{r}\,'(t) = \hat{\imath} + 2t\hat{\jmath}$

$|\vec{r}\,'(t)| = \sqrt{1+4t^2}$

$\hat{T}(t) = \frac{1}{\sqrt{1+4t^2}}(\hat{\imath} + 2t\hat{\jmath})$

$\hat{T}'(t) = \frac{1}{\sqrt{1+4t^2}}(2\hat{\jmath}) + (\hat{\imath}+2t\hat{\jmath})(-\tfrac{1}{2})(8t)(1+4t^2)^{-3/2}$

$\hat{T}'(1) = \frac{2\hat{\jmath}}{\sqrt{5}} + (\hat{\imath}+2\hat{\jmath})(-4)(5)^{-3/2}$

$= 5^{-3/2}\left[-4\hat{\imath} + 2\hat{\jmath}\right]$

$|\hat{T}'(1)| = 5^{-3/2}\sqrt{16+4} = 5^{-3/2}(2)\sqrt{5}$

$\boxed{\hat{N}(1) = \frac{1}{\sqrt{5}}\left[-2\hat{\imath} + \hat{\jmath}\right]}$

5. (6 points) Find $\vec{r}(t)$ if $\frac{d\vec{r}}{dt} = \frac{1}{1+t^2}\hat{\imath} + \frac{1}{t^2}\hat{\jmath} + \frac{1}{t}\hat{k}$ and $\vec{r}(1) = 2\hat{\imath}$.

$\vec{r}(t) = (\tan^{-1} t + c_1)\hat{\imath} + (-\tfrac{1}{t} + c_2)\hat{\jmath} + (\ln t + c_3)\hat{k}$

$\vec{r}(1) = 2\hat{\imath} = (\tfrac{\pi}{4} + c_1)\hat{\imath} + (-1 + c_2)\hat{\jmath} + c_3\hat{k}$

$\Rightarrow c_1 = 2 - \tfrac{\pi}{4}, \quad c_2 = 1, \quad c_3 = 0$

$\boxed{\vec{r}(t) = (\tan^{-1} t + 2 - \tfrac{\pi}{4})\hat{\imath} + (1 - \tfrac{1}{t})\hat{\jmath} + \ln|t|\,\hat{k}}$

6. (6 points) Find the length of one turn of the helix described $\vec{r}(t) = -5\cos t\,\hat{\imath} + 5\sin t\,\hat{\jmath} + 2t\,\hat{k}$.

$\vec{r}\,'(t) = 5\sin t\,\hat{\imath} + 5\cos t\,\hat{\jmath} + 2\hat{k}$

$|\vec{r}\,'(t)| = \sqrt{25 + 4} = \sqrt{29}$

ONE TURN EXTENDS FROM $t = 0$ TO $t = 2\pi$

$\int_0^{2\pi} \sqrt{29}\,dt = \boxed{\sqrt{29}(2\pi) \approx 33.836}$

7. (6 points) Use vectors to show that the points P, Q, and R are collinear.

$$P(10,-2,-1) \qquad Q(37,-47,71) \qquad R(7,3,-9)$$

$$\left.\begin{array}{l} \overrightarrow{PQ} = 27\hat{\imath} - 45\hat{\jmath} + 72\hat{k} \\ \overrightarrow{PR} = -3\hat{\imath} + 5\hat{\jmath} - 8\hat{k} \end{array}\right\}$$

SINCE $\overrightarrow{PQ} = -9\,\overrightarrow{PR}$,

$\overrightarrow{PQ} \parallel \overrightarrow{PR}$.

$\therefore$ THE POINTS ARE COLLINEAR.

8. (6 points) Find a vector-valued function whose graph is the line through the points above.

$$\vec{v} = \overrightarrow{PR} = -3\hat{\imath} + 5\hat{\jmath} - 8\hat{k}$$

$$R\,(7,3,-9)$$

$$x = -3t + 7$$
$$y = 5t + 3$$
$$z = -8t - 9$$

$$\vec{r}(t) = (-3t+7)\,\hat{\imath} + (5t+3)\,\hat{\jmath} + (-8t-9)\,\hat{k}$$

9. (4 points) If the projection of $\vec{u}$ onto $\vec{v}$ has the same magnitude as the projection of $\vec{v}$ onto $\vec{u}$, can you conclude that $|\vec{u}| = |\vec{v}|$?

$$\left|\operatorname{proj}_{\vec{v}} \vec{u}\right| = \underbrace{\left|\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}\right| = \left|\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u}\right|}_{} = \left|\operatorname{proj}_{\vec{u}} \vec{v}\right|$$

$$\frac{|\vec{u}\cdot\vec{v}|}{\sqrt{\vec{v}\cdot\vec{v}}} = \frac{|\vec{u}\cdot\vec{v}|}{\sqrt{\vec{u}\cdot\vec{u}}} \Rightarrow \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\vec{u}\cdot\vec{u}}$$

$$\Rightarrow |\vec{v}| = |\vec{u}|$$

YES!

10. (4 points) Give an equation of the plane passing through $(7, 3, -2)$ and parallel to the plane $6x - 2y + 3z = 9$.

$$6x - 2y + 3z = d$$

$$6(7) - 2(3) + 3(-2) = 42 - 6 - 6 = 30$$

$$\boxed{6x - 2y + 3z = 30}$$

11. (4 points) If $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, is $\vec{u}$ necessarily orthogonal to $2\vec{v} - 5\vec{w}$? Explain.

YES! $\vec{u} \cdot \vec{v} = 0$ AND $\vec{u} \cdot \vec{w} = 0$

$$\Rightarrow \vec{u} \cdot (2\vec{v} - 5\vec{w})$$

THE DOT PRODUCT DISTRIBUTES.

$$= 2\vec{u} \cdot \vec{v} - 5\vec{u} \cdot \vec{w} = 0$$

12. (10 points) A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 meters. Find the required initial velocity and the maximum height of the projectile.

$$\vec{r}(t) = V_0 \cos 8^\circ t\, \hat{i} + (-4.9t^2 + V_0 \sin 8^\circ t)\, \hat{j}$$

$$V_0 \cos 8^\circ t = 50 \Rightarrow t = \frac{50}{V_0 \cos 8^\circ}$$

$$-4.9t^2 + V_0 \sin 8^\circ t = 0$$

$$-4.9t^2 + 50 \tan 8^\circ = 0$$

$$t \approx 1.1975 \Rightarrow \boxed{V_0 = 42.16 \text{ m/sec}}$$

MAX HEIGHT...

$$-9.8t + V_0 \sin 8^\circ = 0$$

$$t = \frac{V_0 \sin 8^\circ}{9.8} \approx 0.5987...$$

AT THAT TIME,

$$-4.9t^2 + V_0 \sin 8^\circ t$$

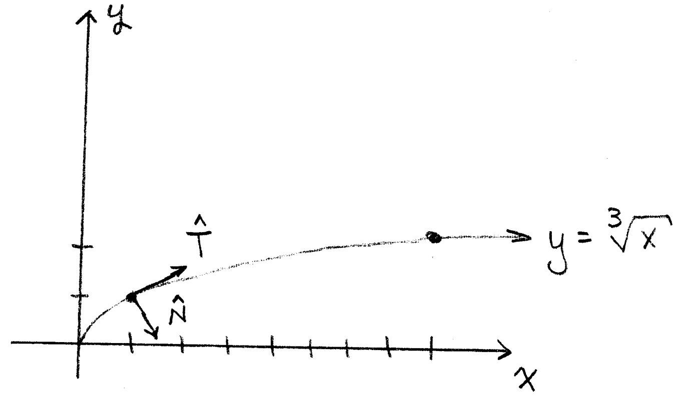
$$= \boxed{1.75676 \text{ m}}$$

13. (8 points) For $t \geq 0$, sketch the graph of $\vec{r}(t) = t^3\hat{\imath} + t\hat{\jmath}$. Without computing them, sketch the unit tangent vector and the principal unit normal vector at the point where $t = 1$.

$\vec{r}(t) = t^3 \hat{\imath} + t\hat{\jmath}$

$x = t^3$
$y = t \Rightarrow x = y^3$

$y = \sqrt[3]{x}$



14. (8 points) Consider the surface described by the equation $y^2 - 4 = 4x^2 + 4z^2$.

(a) What can you say about the level curve at $y = 0$?

$$y = 0 \Rightarrow \underbrace{-4 = 4x^2 + 4z^2}_{\text{Impossible! No such curve.}}$$

(b) Describe the level curve at $y = 3$.

$$y = 3 : \quad 9 - 4 = 4x^2 + 4z^2 \Rightarrow \underbrace{\frac{5}{4} = x^2 + z^2}$$

Circle centered at $(x,z) = (0,0)$ with radius $\frac{\sqrt{5}}{2}$

(c) Describe the level curve at $z = 0$.

$$z = 0 : \quad \underbrace{y^2 - 4x^2 = 4}$$

Hyperbola in xy-plane

(d) Identify the surface.

Hyperboloid of Two Sheets

15. (6 points) Find a point at which the graphs of the following vector-valued functions intersect.

$$\vec{r}(t) = t^2\hat{\imath} + (9t - 20)\hat{\jmath} + t^2\hat{k}$$
$$\vec{s}(t) = (3t + 4)\hat{\imath} + t^2\hat{\jmath} + (5t - 4)\hat{k}$$

MUST HAVE

$$\left.\begin{aligned} t_1^2 &= 3t_2 + 4 \\ 9t_1 - 20 &= t_2^2 \\ t_1^2 &= 5t_2 - 4 \end{aligned}\right\} \quad 3t_2 + 4 = 5t_2 - 4$$

$$8 = 2t_2$$

$$t_2 = 4$$

$$\Downarrow$$

$$t_1 = 4$$

POINT OF INTERSECTION AT $t = 4$:

$(16, 16, 16)$

16. (Extra Credit—5 points) Find the $\underbrace{\text{angle of intersection of the}}_{\text{ANGLE BETWEEN TANGENT VECTORS}}$ two curves above at your point of intersection.

$$\vec{r}\,'(t) = 2t\,\hat{\imath} + 9\hat{\jmath} + 2t\,\hat{k} \qquad \vec{r}\,'(4) = 8\hat{\imath} + 9\hat{\jmath} + 8\hat{k}$$

$$s'(t) = 3\hat{\imath} + 2t\hat{\jmath} + 5\hat{k} \qquad s'(4) = 3\hat{\imath} + 8\hat{\jmath} + 5\hat{k}$$

$$\cos\theta = \frac{8(3) + 9(8) + 8(5)}{\sqrt{64+81+64}\,\sqrt{9+64+25}} = \frac{136}{\sqrt{209 \times 98}}$$

$$\cos\theta = \frac{136}{\sqrt{20482}} \Rightarrow \theta \approx 18.14°$$