Math 173 - Test 2a

March 15, 2012

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Determine the limit or show that it does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy - x}{x^2y^2 + x}$ $\frac{0}{0}$

   $$= \lim_{(x,y)\to(0,0)} \frac{x(y-1)}{x(xy^2+1)} = \lim_{(x,y)\to(0,0)} \frac{y-1}{xy^2+1} = \frac{-1}{1} = \boxed{-1}$$

   (b) $\displaystyle\lim_{(x,y)\to(2,2)} \frac{y - 2}{x^2 - 4}$ $\frac{0}{0}$

   Along $y = 2$: $\displaystyle\lim_{x\to 2} \frac{0}{x^2-4} = \lim_{x\to 2} 0 = 0$

   Along $y = x$: $\displaystyle\lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

   Two different limits along two paths

   $\Longrightarrow$ Limit DNE.

2. (10 points) Consider the graph of the function $f(x, y) = 2x^2 - 4xy^2$. Find parametric equations for the normal line and an equation for the tangent plane at the point $\underbrace{(-1, 2)}$.

$z = 2x^2 - 4xy^2 \Rightarrow F(x,y,z) = 2x^2 - 4xy^2 - z$ $\qquad f(-1,2) = 18$

$F(x,y,z) = 0$ IS THE LEVEL SURFACE THAT <u>IS</u> OUR SURFACE.

$\vec{\nabla} F(x,y,z) = (4x - 4y^2)\hat{\imath} + (-8xy)\hat{\jmath} - \hat{k}$

$\vec{N} = \vec{\nabla} F(-1,2,18) = -20\hat{\imath} + 16\hat{\jmath} - \hat{k}$

NORMAL LINE:

$x = -20t - 1$

$y = 16t + 2$

$z = -t + 18$

PLANE:

$-20(x+1) + 16(y-2) - (z-18) = 0$

OR

$-20x + 16y - z = 34$

3. (5 points) Describe the domain of $f(x, y) = \ln(4 - x - y)$. Then sketch the level curve $f(x, y) = 1$.

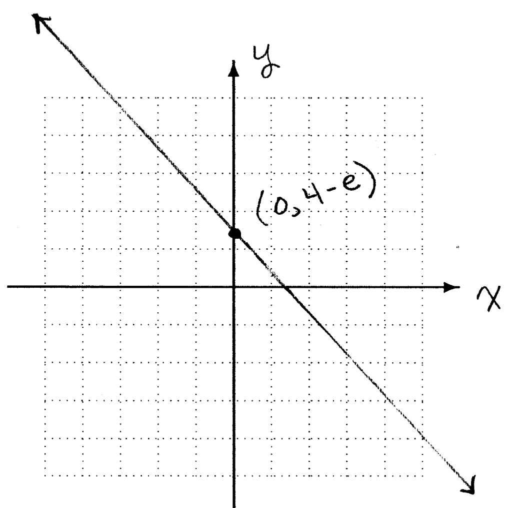
$f(x,y) = \ln(4-x-y)$

MUST HAVE

$4 - x - y > 0$

DOMAIN:

$\{(x,y) : x + y < 4\}$



LEVEL CURVE:

$\ln(4-x-y) = 1$

$\Downarrow$

$4 - x - y = e$

$\therefore y = -x + 4 - e$

$y = -x + 1.2817$

4. (5 points) Suppose z is a function of x, y and x, y are functions of t, u, v. Write the chain rule formulas for $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

5. (8 points) A bug located at $\overset{P}{(3,9,4)}$ begins walking in a straight line toward $\overset{Q}{(5,7,3)}$. At what rate is the bug's temperature changing at the initial point if the temperature is described by $T(x,y,z) = xe^{y-z}$.

$$\vec{PQ} = 2\hat{i} - 2\hat{j} - \hat{k} \qquad |\vec{PQ}| = \sqrt{4+4+1}$$

$$\vec{\nabla} T(x,y,z) = e^{y-z}\hat{i} + xe^{y-z}\hat{j} - xe^{y-z}\hat{k}$$

$$\vec{\nabla} T(3,9,4) = e^{5}\hat{i} + 3e^{5}\hat{j} - 3e^{5}\hat{k}$$

$$D_{\vec{PQ}}\, T(3,9,4) = \frac{1}{|\vec{PQ}|}\vec{\nabla} T(3,9,4)\cdot\vec{PQ}$$

$$= \frac{1}{3}e^{5}(2-6+3) = \boxed{\frac{-e^{5}}{3}}$$

6. (6 points) Let $h(x,y) = \ln(x^3+y^3)$. Find $h_{xy}(x,y)$. Without computing h_{yx}, what can you say about this other mixed partial derivative? Briefly explain.

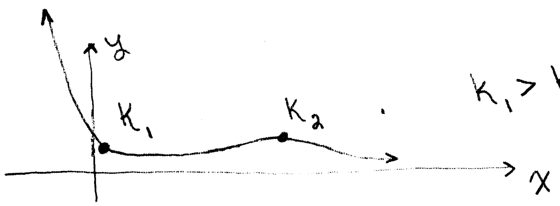
$$h_x(x,y) = \frac{3x^2}{x^3+y^3}$$

$$h_{xy}(x,y) = \frac{-3x^2(3y^2)}{(x^3+y^3)^2} = \frac{-9x^2y^2}{(x^3+y^3)^2}$$

SINCE $h(x,y)$ AND ALL ITS PARTIALS AND MIXED PARTIALS ARE CONTINUOUS WHEN $x^3+y^3>0$, WE EXPECT $h_{xy} = h_{yx}$ BY A THEOREM FROM CLASS.

7. (4 points) Suppose f is a function such that $f''(x)$ exits for all x. Can the curvature function ever be greater than it is at a relative extreme point. Explain.

YES, BUT FOR LOTS OF FUNCTIONS, THE MAX CURVATURE DOES OCCUR AT A RELATIVE MAX / MIN.



8. (4 points) Two cars are racing around a circular track. At a certain moment, both speedometers read 120 mph. Which component of acceleration, tangential or normal, is the same for each car at that moment? Explain. (Hint: Refer to the formulas on your laminated sheet.)

THE CURVATURE IS THE SAME FOR BOTH BECAUSE THEY ARE ON A CIRCULAR TRACK. THE SPEED IS SAME FOR BOTH (I.E. 120 MPH). THEREFORE

$\underbrace{a_N = k|\vec{v}|^2}_{\text{NORMAL COMPONENT}}$ IS THE SAME FOR BOTH.

9. (8 points) Find and classify the critical points of $g(x, y) = x^2 - xy + 2y^2 + 7x$.

$g_x(x,y) = 2x - y + 7$

$g_y(x,y) = -x + 4y$

$g_{xx}(x,y) = 2$

$g_{yy}(x,y) = 4$

$g_{xy}(x,y) = -1$

ONLY CRIT PTS OCCUR WHERE

$$\begin{array}{r} 2x - y = -7 \\ +\,2(-x + 4y = 0) \\ \hline 7y = -7 \\ y = -1 \\ x = -4 \end{array}$$

$$d = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} = 7$$

AT $(-4,-1)$,
$d > 0$ AND $f_{xx} > 0$

$\Rightarrow$ $\boxed{\text{REL MIN AT } (-4,-1) \text{ WHERE } f(-4,-1) = -14}$

Math 173 - Test 2b

March 15, 2012

Name key

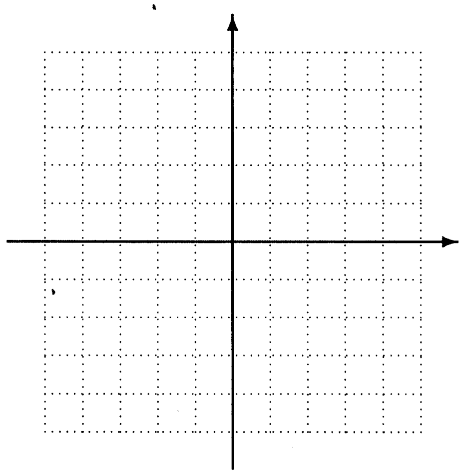
Score

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Calculate the curvature $\kappa(t)$ of the twisted cubic $\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k}$. Then sketch the graph of $\kappa(t)$ (You may use your graphing calculator or a CAS.) and use the graph to determine the point at which the curvature is greatest.

$$\vec{r}\,'(t) = \hat{\imath} + 2t\hat{\jmath} + 3t^2\hat{k}$$

$$\vec{r}\,''(t) = 2\hat{\jmath} + 6t\hat{k}$$



$$K(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\hat{\imath} - 6t\hat{\jmath} + 2\hat{k}$$

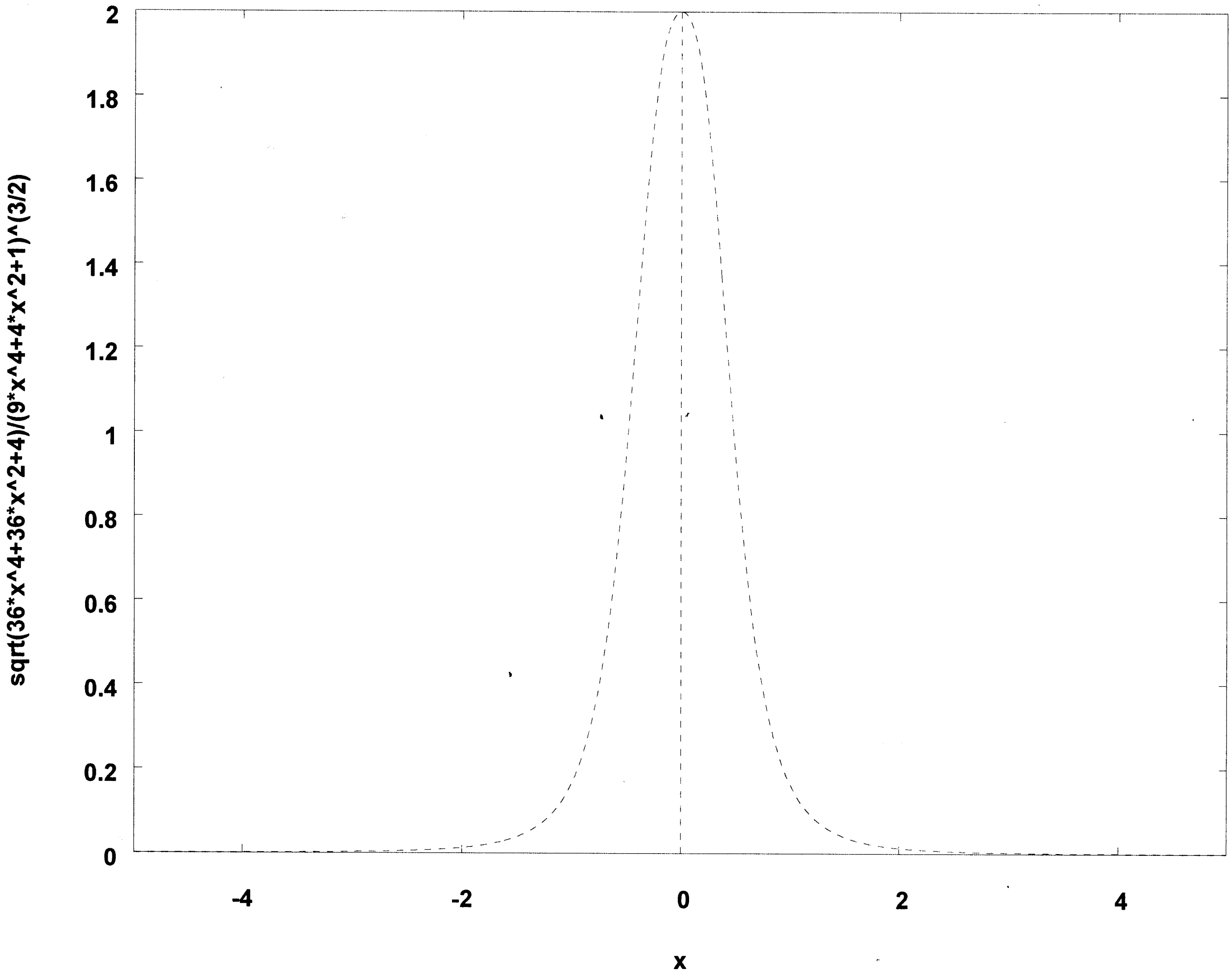
$$\boxed{K(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3}}$$

SEE ATTACHED SHEET FOR graph.

MAX $K(t)$ OCCURS WHEN $t = 0$:

$\vec{r}(0) = \vec{0}$

$\Rightarrow$ GREATEST CURVATURE AT $(0,0,0)$.



sqrt(36*x^4+36*x^2+4)/(9*x^4+4*x^2+1)^(3/2)
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-4
-2
0
2
4
x

2. (10 points) The linearization of a function $F(x, y)$ at the point (x_0, y_0) is the function

$$L(x, y) = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0).$$

Suppose we are given the two functions

$$f(x, y) = 4x^2 + y^2 - 4, \qquad g(x, y) = x + y - \sin(x - y)$$

and we wish to solve the system of nonlinear equations $f(x, y) = g(x, y) = 0$. We can approximate a solution using linearizations and Newton's method.

(a) Let $(x_0, y_0) = (1, 0)$ be our initial guess at the solution. Let $f_0(x, y)$ and $g_0(x, y)$ be the linearizations for f and g at the point (x_0, y_0). Find $f_0(x, y)$ and $g_0(x, y)$.

$f_x(x,y) = 8x$

$f_y(x,y) = 2y$

$g_x(x,y) = 1 - \cos(x-y)$

$g_y(x,y) = 1 + \cos(x-y)$

$f_0(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)\,y$

$\boxed{f_0(x,y) = 8(x-1)}$

$g_0(x,y) = g(1,0) + g_x(1,0)(x-1) + g_y(1,0)\,y$

$g_0(x,y) = (1 - \sin 1) + (1 - \cos 1)(x-1) + (1 + \cos 1)y$

(b) Solve the linear system of equations

$$f_0(x, y) = 0, \qquad g_0(x, y) = 0.$$

$8(x-1) = 0 \Rightarrow \boxed{x = 1}$

SUBS

$(1 - \sin 1) + (1 - \cos 1)(x-1) + (1 + \cos 1)y = 0$

$\Rightarrow y = \dfrac{\sin(1) - 1}{1 + \cos(1)} \approx -0.10292$

(c) Let (x_1, y_1) be the solution of the linear system in part (b). It represents an improved guess at the solution. Compute $f(x_1, y_1)$ and $g(x_1, y_1)$.

$f(x_1, y_1) = 0.01059267\ldots \qquad g(x_1, y_1) = 0.00455090\ldots$

(d) (Extra Credit 4 pts) On a separate sheet of paper, use (x_1, y_1) in place of (x_0, y_0) and repeat the steps above to further improve our guess at the solution.

SEE ATTACHED SHEETS.

$x_2 \approx 0.9986$

$y_2 \approx -0.1055$

```
(%i1) f: 4*x^2+y^2-4; g: x+y-sin(x-y);
(%o1) y^2 +4 x^2 -4
(%o2) sin(y-x)+y+x

(%i3) x1: float( 1 ); y1: float( (sin(1)-1)/(1+cos(1)) );
(%o3) 1.0
(%o4) -0.10292071536097

(%i5) fx: diff(f,x); fy: diff(f,y);
(%o5) 8 x
(%o6) 2 y

(%i7) gx: diff(g,x); gy: diff(g,y);
(%o7) 1-cos(y-x)
(%o8) cos(y-x)+1

(%i9) f2: at(f,[x=x1,y=y1]) + at(fx,[x=x1,y=y1])*(x-x1) + at(fy,[x=x1,y=
(%o9) -0.20584143072194 (y+0.10292071536097 )+8.0 (x-1.0)+
0.010592673650414

(%i10) expand(f2);
(%o10) -0.20584143072194 y+8.0 x-8.010592673650415

(%i11) g2: at(g,[x=x1,y=y1]) + at(gx,[x=x1,y=y1])*(x-x1) + at(gy,[x=x1,y=
(%o11) 1.450991227382808 (y+0.10292071536097 )+0.54900877261719
(x-1.0)+0.0045509025559998

(%i12) expand(g2);
(%o12) 1.450991227382808 y+0.54900877261719 x-0.39512081495646

(%i13) float(solve([f2=0,g2=0],[x,y]));
rat: replaced 0.010592673650414 by 657/62024 = 0.010592673803689
rat: replaced 8.0 by 8/1 = 8.0
rat: replaced -1.0 by -1/1 = -1.0
rat: replaced -0.2058414307219 by -18197/88403 = -0.2058414307207
rat: replaced 0.10292071536097 by 18197/176806 = 0.10292071536034
rat: replaced 0.0045509025559998 by 505/110967 = 0.0045509025205692
rat: replaced 0.54900877261719 by 7948/14477 = 0.54900877253575
rat: replaced -1.0 by -1/1 = -1.0
rat: replaced 1.450991227382808 by 21006/14477 = 1.450991227464254
rat: replaced 0.10292071536097 by 18197/176806 = 0.10292071536034
(%o13) [[x=0.99860875978901 , y=-0.10553072386235 ]]
```

```
(%i14) x2: 0.99860875978901; y2: -0.10553072386235;
(%o14) 0.99860875978901
(%o15) -0.10553072386235
```

```
(%i16) at(f,[x=x2,y=y2]); at(g,[x=x2,y=y2]);
```

(%o16) $1.4554188290237524\ 10^{-5}$

(%o17) $6.6305155710377761\ 10^{-7}$

3. (10 points) Find and classify the critical points of $f(x, y) = y^2x - yx^2 + xy$.

$f_x(x,y) = y^2 - 2xy + y = y(y-2x+1)$

$f_y(x,y) = 2xy - x^2 + x = x(2y-x+1)$

$y(y-2x+1) = 0$

$x(2y-x+1) = 0$

$\Rightarrow$ 4 CASES :

① $x = 0,\ y = 0$

② $y = 0,\ 2y - x + 1 = 0$

$\Rightarrow x = 1,\ y = 0$

③ $x = 0,\ y - 2x + 1 = 0$

$\Rightarrow x = 0,\ y = -1$

④ $y - 2x + 1 = 0$
$2y - x + 1 = 0$

$\longrightarrow$

$$\begin{array}{r} -2y + 4x - 2 = 0 \\ 2y - x + 1 = 0 \\ \hline 3x - 1 = 0 \end{array}$$

$x = \frac{1}{3},\ y = -\frac{1}{3}$

$f_{xx}(x,y) = -2y$

$f_{yy}(x,y) = 2x$

$f_{xy}(x,y) = 2y - 2x + 1$

$$d(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$$= -4xy - (2y - 2x + 1)^2$$

① $d(0,0) = -1 \Rightarrow (0,0,0)$ IS A SADDLE PT.

② $d(1,0) = -1 \Rightarrow (1,0,0)$ IS A SADDLE PT.

③ $d(0,-1) = -1 \Rightarrow (0,1,0)$ IS A SADDLE PT.

④ $d(\frac{1}{3}, -\frac{1}{3}) = \frac{1}{3}$ AND $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = \frac{2}{3} \Rightarrow f(\frac{1}{3}, -\frac{1}{3}) = -\frac{1}{27}$ IS A RELATIVE MIN.

4. (8 points) Suppose that $y = \left(\dfrac{2L^2 v_0^2}{3g}\right)^{1/3}$. Use differentials to estimate the progagated error, Δy, if the measured values of L, v_0, and g are 1.546, 0.84, and 9.78, respectively, and the measurement errors are $\Delta L = 0.005$, $\Delta v_0 = 0.1$, and $\Delta g = 0.003$.

$$\Delta y \approx \frac{\partial y}{\partial L}\Delta L + \frac{\partial y}{\partial V_0}\Delta V_0 + \frac{\partial y}{\partial g}\Delta g$$

$$\Delta y \approx \frac{1}{3}\left(\frac{2L^2V_0^2}{3g}\right)^{-2/3}\left[\frac{4LV_0^2}{3g}\Delta L + \frac{4L^2V_0}{3g}\Delta V_0 - \frac{2L^2V_0^2}{3g^2}\Delta g\right]$$

NOW SUBSTITUTE THE VALUES OF $L, V_0, g, \Delta L, \Delta V_0, \Delta g$

TO GET

$$\Delta y \approx 0.0396$$

SEE ATTACHED SHEET.

5. (4 points) If an object's speed is constant, what can be said about its tangential component of acceleration? Use a formula for a_T (see page 877) to argue that, in this case, $\vec{v}$ and $\vec{a}$ are orthogonal.

IF SPEED IS CONSTANT,

$a_T = 0$.

$a_T = \dfrac{\vec{V}\cdot\vec{a}}{|\vec{V}|}$; IF THIS IS ZERO, $\vec{V}\cdot\vec{a} = 0$

AND THEREFORE $\vec{V}$ AND $\vec{a}$ ARE ORTHOGONAL.

(%i1) `y: (2*L^2*v0^2/(3*g))^(1/3);`

(%o1) $$\frac{2^{1/3}\, v0^{2/3}\, L^{2/3}}{3^{1/3}\, g^{1/3}}$$

(%i2) `dy: diff(y,L)*dL+diff(y,v0)*dv0+diff(y,g)*dg;`

(%o2) $$-\frac{2^{1/3}\, dg\, v0^{2/3}\, L^{2/3}}{3^{4/3}\, g^{4/3}}+\frac{2^{4/3}\, dv0\, L^{2/3}}{3^{4/3}\, g^{1/3}\, v0^{1/3}}+\frac{2^{4/3}\, v0^{2/3}\, dL}{3^{4/3}\, g^{1/3}\, L^{1/3}}$$

(%i4) `float(at(dy, [L=1.546,v0=0.84,g=9.78,dL=0.005,dv0=0.1,dg=0.003]));`

(%o4) 0.039589001221331