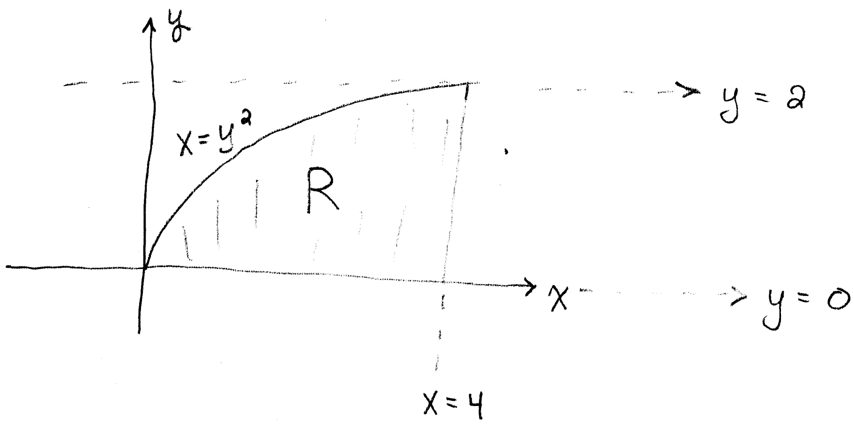
Math 173 - Test 3
April 26, 2012

Name key
Score

Show all work to receive full credit. Supply explanations where necessary. You must work individually on this exam. Unless otherwise indicated, do all integration by hand, without the help of a CAS.

1. (15 points) Sketch the region of integration, reverse the order, and evaluate. Do not use your calculator.

$$\int_0^2 \int_{y^2}^4 \sqrt{x}\, \sin x \, dx \, dy$$



$$\int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{x}} \sqrt{x}\, \sin x \, dy \, dx$$

$$= \int_0^4 x \sin x \, dx$$

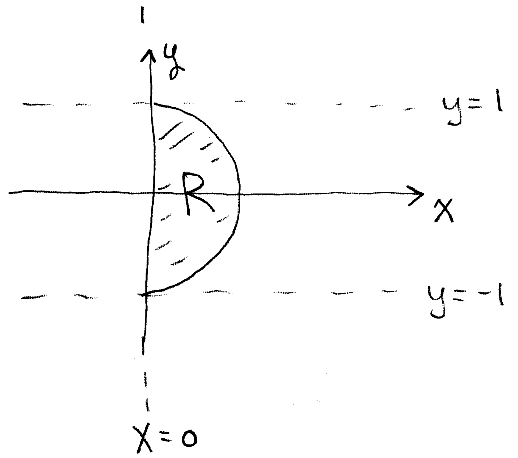
+	x	sin x
−	1	−cos x
+	0	−sin x

$$= -x \cos x + \sin x \Big|_0^4$$

$$= \boxed{\sin 4 - 4\cos 4}$$

$x=\sqrt{1-y^2} \Rightarrow x^2+y^2=1$

2. (15 points) Convert $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$ to an integral in cylindrical coordinates. Then evaluate.



$$\iint_R \int_0^x (x^2+y^2)\,dz\,dA$$

$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} \int_{z=0}^{z=r\cos\theta} r^3\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^1 r^4\cos\theta\,dr\,d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta$$

$$= \frac{2}{5}\int_0^{\pi/2}\cos\theta\,d\theta = \boxed{\frac{2}{5}}$$

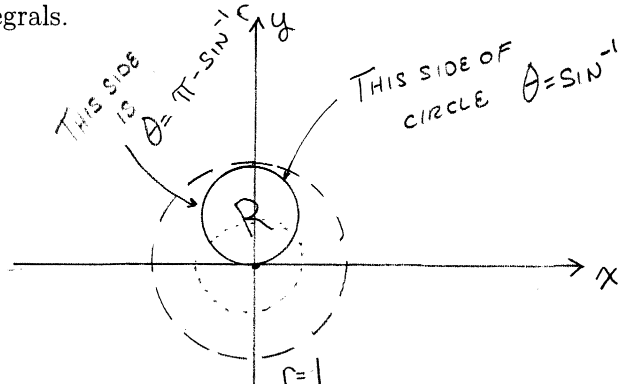
3. (12 points) Consider the following iterated integral in polar coordinates.

$$\int_0^{\pi}\int_0^{\sin\theta} r^2\,dr\,d\theta$$

Sketch the region of integration. Then write a new iterated integral with the order of integration reversed. Finally convert to an iterated integral in rectangular coordinates. You need not evaluate any of the integrals.

R: $0 \le \theta \le \pi$

$0 \le r \le \sin\theta$



$r=\sin\theta \Rightarrow r^2 = r\sin\theta$

$\Rightarrow x^2+y^2=y \Rightarrow x^2+y^2-y=0$

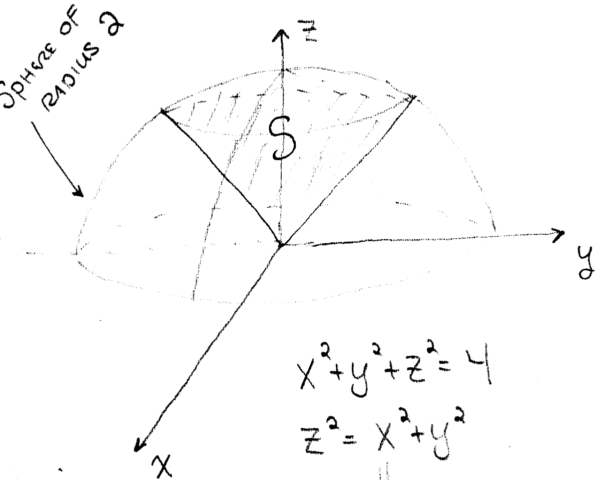
$\Rightarrow x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$ CIRCLE AT $(0,\frac{1}{2})$ w/ RADIUS $1/2$

$$\boxed{\int_{r=0}^{r=1}\int_{\theta=\sin^{-1}r}^{\theta=\pi-\sin^{-1}r} r^2\,d\theta\,dr}$$

$$\int_{y=0}^{y=1}\int_{x=-\sqrt{y-y^2}}^{x=\sqrt{y-y^2}} \sqrt{x^2+y^2}\,dx\,dy$$

OR $\displaystyle 2\int_0^1\int_{\sin^{-1}r}^{\pi/2} r^2\,d\theta\,dr$

4. (15 points) Set up the triple integral required to find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$. Use a CAS to evaluate the integral.



$$V = \iiint_S dV$$

$$= \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \frac{16\pi}{3} - \frac{8\sqrt{2}\,\pi}{3}$$

$$\approx 4.907$$

$x^2+y^2+z^2=4$

$z^2 = x^2+y^2$

$\Downarrow$

$2z^2 = 4$

$z = \sqrt{2}$

$\rho \cos\varphi = z \Rightarrow \cos\varphi = \frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{\pi}{4}$

5. (15 points) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = y^2 - 4x$ on the circle $x^2 + y^2 = 9$.

$f(x,y) = y^2 - 4x$

$g(x,y) = x^2 + y^2$

$\Rightarrow \vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow$

$-4 = 2\lambda x$

$2y = 2\lambda y$

ALSO, $x^2+y^2=9$

$2y - 2\lambda y = 2y(1-\lambda) = 0$

$\Rightarrow y = 0$ OR $\lambda = 1$

$\underline{\lambda = 1}$

$x = -2$

$(-2)^2 + y^2 = 9$

$y = \pm\sqrt{5}$

$(-2, \sqrt{5}), (-2, -\sqrt{5})$

$\underline{y = 0}$

$x^2 = 9$

$x = \pm 1$

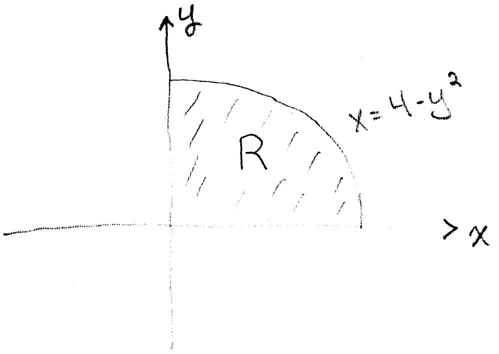
$(3,0)$, $(-3,0)$

$$\boxed{\begin{array}{l} f(-2,\sqrt{5}) = 13 \\ f(-2,-\sqrt{5}) = 13 \end{array}} \Longrightarrow \text{MAX VALUE IS } 13$$

$$\boxed{f(3,0) = -12} \Longrightarrow \text{MIN VALUE IS } -12.$$

$f(-3,0) = 12$

6. (16 points) A solid lies in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the parabolic cylinder $x = 4 - y^2$. The density of the solid at the point (x, y, z) is given by $\delta(x, y, z) = 2 + x + y + z$. Set up the integrals required to determine the center of mass. Use a CAS to evaluate the integrals and find the center of mass.



THE PLANE $y + z = 2$ OR $z = 2 - y$ LIES OVER THE REGION R.

THE SOLID IS BOUNDED ABOVE BY $z = 2 - y$ AND HAS R AS THE BASE OF THE REGION.

$$M = \int_{y=0}^{y=2} \int_{x=0}^{x=4-y^2} \int_{z=0}^{z=2-y} (2 + x + y + z)\, dz\, dx\, dy = \frac{168}{5}$$

$$M_{yz} = \int_{y=0}^{y=2} \int_{x=0}^{x=4-y^2} \int_{z=0}^{z=2-y} x(2 + x + y + z)\, dz\, dx\, dy = \frac{6976}{105}$$

$$M_{xz} = \int_{y=0}^{y=2} \int_{x=0}^{x=4-y^2} \int_{z=0}^{z=2-y} y(2 + x + y + z)\, dz\, dx\, dy = \frac{1952}{105}$$

$$M_{xy} = \int_{y=0}^{y=2} \int_{x=0}^{x=4-y^2} \int_{z=0}^{z=2-y} z(2 + x + y + z)\, dz\, dx\, dy = \frac{8048}{315}$$

CENTER OF MASS =

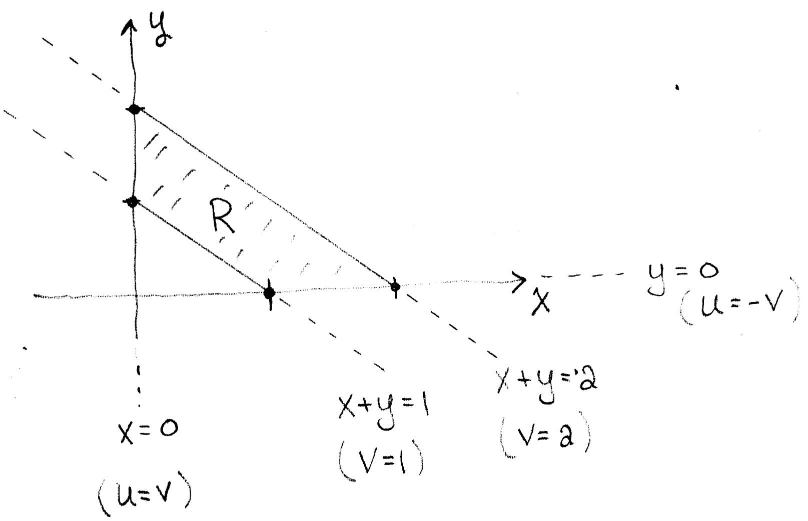
$$\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right) = \left(\frac{872}{441}, \frac{244}{441}, \frac{1006}{1323} \right)$$

$$\approx (1.98, 0.55, 0.76)$$

7. (12 points) By making an appropriate change of variables, evaluate

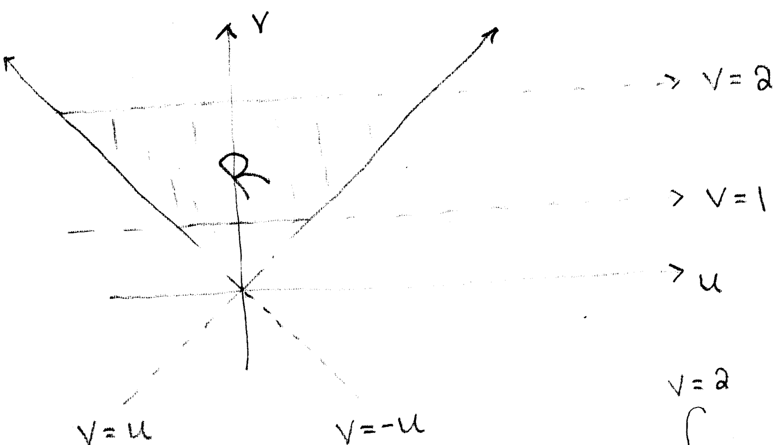
$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA,$$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.



Let $u = y - x$, $v = y + x$ $\Rightarrow$ $y = \dfrac{u+v}{2}$, $x = \dfrac{v-u}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$



$$\int_{v=1}^{v=2} \int_{u=-v}^{u=v} \cos\left(\frac{u}{v}\right)\left(\frac{1}{2}\right) du\, dv$$

$$= \frac{1}{2}\int_1^2 v \sin\left(\frac{u}{v}\right)\Bigg|_{u=-v}^{u=v} dv$$

$$= \sin(1)\int_1^2 v\, dv = \sin(1)\,\frac{v^2}{2}\Bigg|_1^2 = \sin(1)\left(2 - \frac{1}{2}\right) = \boxed{\frac{3}{2}\sin(1)}$$

≈ 1.26