Math 173 - 1st Final Exam

May 3, 2012

Name key

Score

Show all work. Supply explanations when necessary. **Unless otherwise specified, you may use your calculator to evaluate any integrals.** Each problem is worth 10 points.

1. Find $\vec{r}(t)$ given that $\underbrace{\frac{d\vec{r}}{dt} = \frac{1}{1+t^2}\hat{\imath} + \frac{1}{t^2}\hat{\jmath} + \frac{1}{t}\hat{k}}$ and $\vec{r}(1) = 2\hat{\imath}$.

$$\vec{r}(t) = \left(\tan^{-1} t + c_1\right)\hat{\imath} + \left(-\frac{1}{t} + c_2\right)\hat{\jmath} + \left(\ln|t| + c_3\right)\hat{k}$$

$$\vec{r}(1) = \left(\frac{\pi}{4} + c_1\right)\hat{\imath} + \left(-1 + c_2\right)\hat{\jmath} + c_3\hat{k}$$

$$= 2\hat{\imath} \Rightarrow c_1 = 2 - \frac{\pi}{4}, \quad c_2 = 1, \quad c_3 = 0$$

$$\boxed{\vec{r}(t) = \left(2 - \frac{\pi}{4} + \tan^{-1} t\right)\hat{\imath} + \left(-\frac{1}{t} + 1\right)\hat{\jmath} + \ln|t|\,\hat{k}}$$

2. Find a vector of magnitude 5 that is orthogonal to both $\vec{x} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{y} = -\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$.

$$\vec{X} \times \vec{Y} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 1 \\ -1 & 5 & -3 \end{vmatrix} = \hat{\imath}(6-5) - \hat{\jmath}(-9+1) + \hat{k}(15-2)$$

$$= \hat{\imath} + 8\hat{\jmath} + 13\hat{k}$$

$$\frac{5}{|\vec{X} \times \vec{Y}|}\left(\vec{X} \times \vec{Y}\right) = \frac{5}{\sqrt{1+64+169}}\left(\hat{\imath} + 8\hat{\jmath} + 13\hat{k}\right)$$

$$= \boxed{\frac{5}{\sqrt{234}}\left(\hat{\imath} + 8\hat{\jmath} + 13\hat{k}\right)}$$

3. Find the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y}{x+y}$

ALONG $y=0$: $\displaystyle\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$

ALONG $x=0$: $\displaystyle\lim_{y\to 0} \frac{y}{y} = \lim_{y\to 0} 1 = 1$

TWO DIFFERENT LIMITS ALONG TWO DIFFERENT PATHS
$\Rightarrow$ LIMIT DNE

(b) $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x-y}{x^2-y^2} = \lim_{(x,y)\to(2,2)} \frac{x-y}{(x-y)(x+y)} = \lim_{(x,y)\to(2,2)} \frac{1}{x+y}$

$= \boxed{\frac{1}{4}}$

4. Find a set of parametric equations for the line tangent to the graph of $\vec{r}(t)$ at the point $(e, 0, 2)$.

$$\vec{r}(t) = te^t\hat{\imath} + \sin(\pi t)\hat{\jmath} + \sqrt{3+t^2}\hat{k}$$

$r'(t) = (te^t + e^t)\hat{\imath} + \pi\cos(\pi t)\hat{\jmath} + \frac{1}{2}(3+t^2)^{-1/2}(2t)\hat{k}$

$(e,0,2)$ CORRESPONDS TO $t=1$
i.e. $\vec{r}(1) = e\hat{\imath} + 2\hat{k}$

DIRECTION VECTOR: $\vec{r}\,'(1) = 2e\hat{\imath} - \pi\hat{\jmath} + \frac{1}{2}\hat{k}$

$x = 2et + e$
$y = -\pi t$
$z = \frac{1}{2}t + 2$

5. Find the directional derivative of $g(x, y, z) = xye^z$ at $(2, 4, 0)$ in the direction of $(0, 0, 0)$.

$\vec{v}$ = VECTOR FROM $(2,4,0)$ TO $(0,0,0)$ $= -2\hat{i} - 4\hat{j}$, $|\vec{v}| = \sqrt{4+16} = \sqrt{20}$

$$\vec{\nabla} g(x,y,z) = ye^z\,\hat{i} + xe^z\,\hat{j} + xye^z\,\hat{k}$$

$$\vec{\nabla} g(2,4,0) = 4\hat{i} + 2\hat{j} + 8\hat{k}$$

$$D_{\vec{v}}\, g(2,4,0) = \frac{1}{|\vec{v}|}\vec{\nabla} g(2,4,0)\cdot\vec{v} = \frac{1}{\sqrt{20}}(-8-8+0) = \boxed{\frac{-16}{\sqrt{20}}}$$

Follow-up: At the point $(2, 4, 0)$, in what direction is g increasing most rapidly?

IN THE DIRECTION OF $\vec{\nabla} g(2,4,0) = 4\hat{i} + 2\hat{j} + 8\hat{k}$.

6. Use the chain rule to find $\dfrac{\partial w}{\partial s}$ when $s = 4$ and $t = \pi/4$.

$$w = 5x^3 - xy^2; \qquad x = s\cos t, \quad y = s\sin t$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= (15x^2 - y^2)(\cos t) + (-2xy)(\sin t)$$

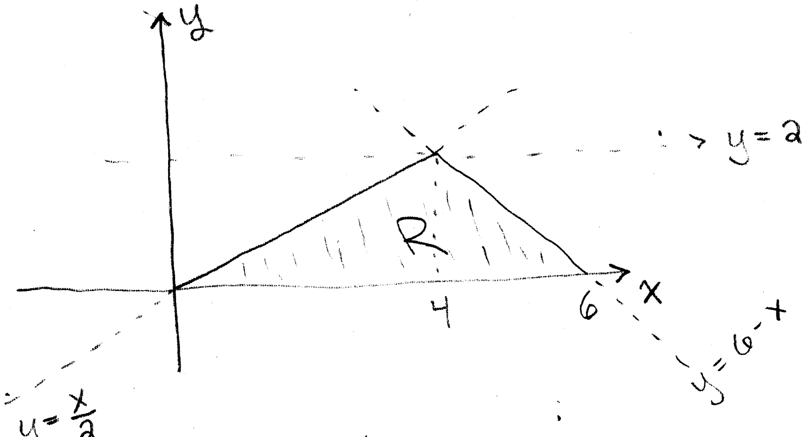
WHEN $S = 4$ AND $t = \pi/4$, $x = 4\cos\frac{\pi}{4} = 2\sqrt{2}$

$y = 4\sin\frac{\pi}{4} = 2\sqrt{2}$

$$\left.\frac{\partial w}{\partial s}\right|_{(4,\pi/4)} = \underline{48\sqrt{2}}$$

7. Sketch the region R whose area is given by the iterated integral. Then reverse the order of integration and evaluate the new iterated integral by hand.

$$\int_0^4 \int_0^{x/2} dy\,dx + \int_4^6 \int_0^{6-x} dy\,dx$$



$$\int_{y=0}^{y=2} \int_{x=2y}^{x=6-y} dx\,dy = \int_0^2 (6-y-2y)\,dy = \int_0^2 (6-3y)\,dy$$

$$= 6y - \frac{3}{2}y^2 \Big|_0^2$$

$$= 12 - 6 = \boxed{6}$$

8. Find and classify all critical points of the function $f(x, y)$.

$$f(x, y) = -x^2 - 5y^2 + 10x - 10y - 28$$

$$f_x(x,y) = -2x + 10 = 0 \qquad x = 5$$

$$\Rightarrow$$

$$f_y(x,y) = -10y - 10 = 0 \qquad y = -1$$

$$d(5,-1) = \begin{vmatrix} -2 & 0 \\ 0 & -10 \end{vmatrix} = 20 > 0$$

$d > 0$ AND $f_{xx} = -2 < 0 \Rightarrow (5,-1)$ GIVES A RELATIVE MAX

$f(5,-1) = 2$ IS A REL MAX.

9. Find the angle between the planes (i.e. the angle between the normal vectors).

$$x - 3y + 6z = 4$$
$$5x + y - z = 4$$

$$\vec{N}_1 = \hat{i} - 3\hat{j} + 6\hat{k}$$
$$\vec{N}_2 = 5\hat{i} + \hat{j} - \hat{k}$$

$$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|} = \frac{5-3-6}{\sqrt{46}\sqrt{27}} = \frac{-4}{\sqrt{1242}}$$

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{1242}}\right) \approx 1.68 \approx 96.5^\circ$$

OR $180^\circ - 96.5^\circ = 83.5^\circ$

IF YOU LOOK AT THE OTHER ANGLE.

10. Find an equation of the plane tangent to the graph of

$$f(x, y) = (x + 1)^2 e^{2y} + y\sin(x + \pi/2)$$

at the point where $(x, y) = (0, 0)$.

$f(0,0) = 1$

$$z = (x+1)^2 e^{2y} + y\sin(x + \pi/2)$$

$$F(x,y,z) = (x+1)^2 e^{2y} + y\sin\left(x + \frac{\pi}{2}\right) - z$$

OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 0$ THROUGH $(0,0,1)$.

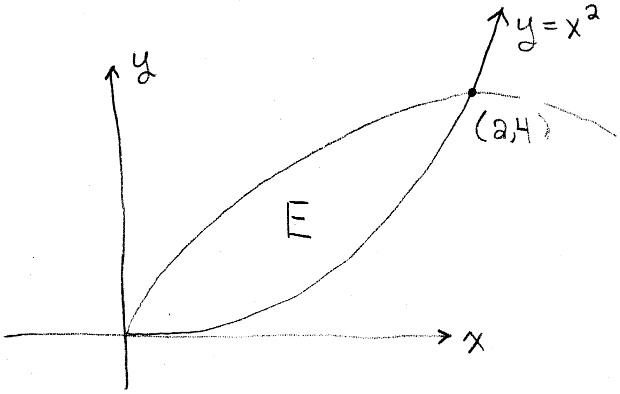
$$\vec{\nabla} F(x,y,z) = \left[2(x+1)e^{2y} + y\cos\left(x + \frac{\pi}{2}\right)\right]\hat{i} + \left[2(x+1)^2 e^{2y} + \sin(x + \pi/2)\right]\hat{j} - \hat{k}$$

$$\vec{\nabla} F(0,0,1) = 2\hat{i} + 3\hat{j} - \hat{k} = \text{NORMAL VECTOR}$$

PLANE: $2(x-0) + 3(y-0) - (z-1) = 0$

OR $\boxed{2x + 3y - z = -1}$

11. Let E be the bounded region between the graphs of $y = 4x - x^2$ and $y = x^2$. Find the average value of $f(x, y) = x + y$ over E.



$$\text{AREA OF } E = \int_{x=0}^{x=2} \int_{y=x^2}^{y=4x-x^2} dy\, dx = \frac{8}{3}$$

$$\text{AVERAGE VALUE} = \frac{1}{\text{AREA OF } E} \int_{x=0}^{x=2} \int_{y=x^2}^{y=4x-x^2} (x+y)\, dy\, dx$$

$$= \frac{1}{8/3}(8) = \boxed{3}$$

12. Find the unit tangent vector at the point where $t = \pi/2$.

$$\vec{r}(t) = 2\sin t\hat{\imath} + 2\cos t\hat{\jmath} + 4\sin^2 t\hat{k}$$

$$\vec{r}\,'(t) = 2\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath} + 8\sin t\cos t\,\hat{k}$$

$$\vec{r}\,'(\pi/2) = 0\hat{\imath} - 2\hat{\jmath} + 0\hat{k}, \quad |\vec{r}\,'(\pi/2)| = 2$$

$$\hat{T}(\pi/2) = \frac{\vec{r}\,'(\pi/2)}{|\vec{r}\,'(\pi/2)|} = \boxed{-\hat{\jmath}}$$

13. Consider the surface described by the equation $z = x^2 + y^2 + 3$.

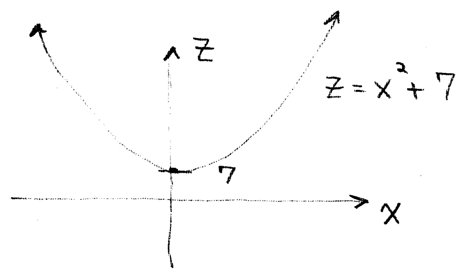
(a) Sketch or describe (in detail) the level curve $z = 4$.

$$z = 4 \Rightarrow 4 = x^2 + y^2 + 3 \quad \text{OR} \quad \underbrace{x^2 + y^2 = 1}_{\text{UNIT CIRCLE CENTERED AT ORIGIN.}}$$

(b) Sketch or describe (in detail) the level curve $y = 2$.

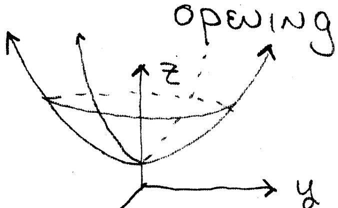
$y = 2$: $\quad z = x^2 + 4 + 3$

$$\underbrace{z = x^2 + 7}_{\text{PARABOLA IN } XZ \text{ WITH VERTEX AT } (0,7)}$$



(c) Sketch the surface.

THE SURFACE IS A CIRCULAR PARABOLOID OPENING UP THE Z-AXIS WITH VERTEX AT $(0,0,3)$



14. Let $\vec{u}$ be the vector from $(6, 3, 1)$ to $(8, 0, 4)$. Let $\vec{v}$ be the vector in the xy-plane with magnitude 4 that makes an angle of 30° with the positive x-axis. Find $\text{proj}_{\vec{u}}\, \vec{v}$.

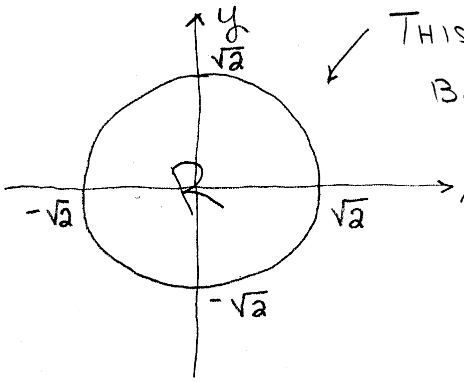
$$\vec{u} = 2\hat{\imath} - 3\hat{\jmath} + 3\hat{k}$$

$$\vec{v} = 4\cos 30^\circ\, \hat{\imath} + 4\sin 30^\circ\, \hat{\jmath} = 2\sqrt{3}\,\hat{\imath} + 2\hat{\jmath}$$

$$\text{Proj}_{\vec{u}}\, \vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{4\sqrt{3} - 6}{4 + 9 + 9}\,\vec{u}$$

$$= \boxed{\frac{2\sqrt{3} - 3}{11}\left(2\hat{\imath} - 3\hat{\jmath} + 3\hat{k}\right)}$$

15. The solid region inside the cylinder $x^2 + y^2 = 2$ is bounded below by the surface $z = 0$ and above by $z = x^2 + y^2 + 3$. The density of the solid at the point (x, y, z) is given by $\rho(x, y, z) = y + z^2 + 1$. Find the mass of the solid. Use your calculator to evaluate the integral.



THIS IS THE BASE OF THE SOLID. IT IS BOUNDED ABOVE BY THE PARABOLOID $z = x^2 + y^2 + 3$

$$\text{MASS} = \iint_R \int_{z=0}^{z=x^2+y^2+3} (y + z^2 + 1)\, dz\, dA$$

CONVERT TO CYLINDRICAL COORDS...

$$\text{MASS} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} \int_{z=0}^{z=r^2+3} (r\sin\theta + z^2 + 1)\, r\, dz\, dr\, d\theta$$

$$= \frac{160\pi}{3}$$