Math 173 - 2nd Final Exam

May 14, 2012

Name key

Score

Show all work. Supply explanations when necessary. **Unless otherwise specified, you may use your calculator or CAS to evaluate any integrals.** Each problem is worth 10 points.

1. Find an equation of the plane passing through the three points $P(2,0,3)$, $Q(5,-2,7)$, and $R(8,3,-2)$.

$\vec{PQ} = 3\hat{i} - 2\hat{j} + 4\hat{k}$

$\vec{PR} = 6\hat{i} + 3\hat{j} - 5\hat{k}$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ 6 & 3 & -5 \end{vmatrix}$$

$$= \hat{i}(-2) - \hat{j}(-39) + \hat{k}(21)$$

$$= -2\hat{i} + 39\hat{j} + 21\hat{k}$$

Point $(2,0,3)$

$\vec{N} = -2\hat{i} + 39\hat{j} + 21\hat{k}$ $\Rightarrow$ $-2(x-2) + 39(y-0) + 21(z-3) = 0$

$$\boxed{-2x + 39y + 21z = 59}$$

2. Find a set of parametric equations for the line passing through the point $(7,3,-1)$ and normal to the plane $8x - 6y + 5z = 15$.

Point $(7,3,-1)$

Direction $8\hat{i} - 6\hat{j} + 5\hat{k}$

$$\boxed{\begin{aligned} x &= 8t + 7 \\ y &= -6t + 3 \\ z &= 5t - 1 \end{aligned}}$$

3. Find the limit or show that it does not exist.

(a) $\lim_{(x,y)\to(0,0)} -\frac{xy^2}{x^2+y^4}$

Along $x=0$: $\lim_{y\to 0}\left(-\frac{0}{y^4}\right) = \lim_{y\to 0} 0 = 0$

Along $x=y^2$: $\lim_{y\to 0}\left(-\frac{y^4}{2y^4}\right) = \lim_{y\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$

Two limits along two paths.

Limit DNE.

(b) $\lim_{(x,y,z)\to(0,0,1)} \frac{zx+zy}{e^z(y^2-x^2)} = \lim_{(x,y,z)\to(0,0,1)} \frac{z(x+y)}{e^z(y-x)(x+y)}$

$$= \underbrace{\lim_{(x,y,z)\to(0,0,1)} \frac{z}{e^z(y-x)}}_{\text{Has form } \frac{1}{e(0)}}$$

Limit DNE.

4. Let $\vec{u}$ be the vector of magnitude 7 that has the direction of $3\hat{\imath}+\hat{\jmath}-\hat{k}$. Let $\vec{w}$ be the unit vector in the direction of $\hat{\imath}+2\hat{\jmath}+3\hat{k}$. Find the projection of $\vec{w}$ onto $\vec{u}$.

$$\vec{u} = \frac{7}{\sqrt{9+1+1}}(3\hat{\imath}+\hat{\jmath}-\hat{k}) = \frac{7}{\sqrt{11}}(3\hat{\imath}+\hat{\jmath}-\hat{k})$$

$$\vec{w} = \frac{1}{\sqrt{1+4+9}}(\hat{\imath}+2\hat{\jmath}+3\hat{k}) = \frac{1}{\sqrt{14}}(\hat{\imath}+2\hat{\jmath}+3\hat{k})$$

$$\text{proj}_{\vec{u}}\,\vec{w} = \frac{\vec{u}\cdot\vec{w}}{\vec{u}\cdot\vec{u}}\,\vec{u} = \frac{\frac{7}{\sqrt{154}}(3+2-3)}{\frac{49}{11}(9+1+1)}\left(\frac{7}{\sqrt{11}}\right)(3\hat{\imath}+\hat{\jmath}-\hat{k})$$

2

$$= \boxed{\frac{2}{11\sqrt{14}}(3\hat{\imath}+\hat{\jmath}-\hat{k})}$$

5. The quarterback of a football team releases a pass at a height of 7 ft above the playing field, and the football is caught by a receiver 90 ft directly downfield at a height of 4 ft. The pass is released at an angle of 35° with the horizontal. Find the speed of the football when it is released.

$$\vec{r}(t) = (V_0 \cos\theta \, t + x_0)\hat{\imath} + \left(-\tfrac{1}{2} g t^2 + V_0 \sin\theta \, t + y_0\right)\hat{\jmath}$$

$$\vec{r}(t) = (V_0 \cos 35° \, t)\hat{\imath} + (-16t^2 + V_0 \sin 35° \, t + 7)\hat{\jmath}$$

$$V_0 \cos 35° \, t = 90 \Rightarrow V_0 t = \frac{90}{\cos 35°}$$

$$-16t^2 + V_0 \sin 35° t + 7 = 4 \longrightarrow -16t^2 + 90 \tan 35° + 7 = 4$$

$$t^2 = \frac{4 - 7 - 90 \tan 35°}{-16} \approx 4.126$$

$$t \approx 2.0313 \text{ SEC}$$

$$V_0 = \frac{90}{2.0313(\cos 35°)} \approx \boxed{54.088 \text{ FT/S}}$$

6. A surface is described by the equation $xy^2 + 3x - z^2 = 8$. Find an equation of the plane tangent to the surface at the point $(1, -3, 2)$.

$$F(x,y,z) = xy^2 + 3x - z^2$$

OUR SURFACE IS THE LEVEL SURFACE $F(x,y,z) = 8$.

$$\vec{\nabla} F(x,y,z) = (y^2+3)\hat{\imath} + 2xy\,\hat{\jmath} - 2z\,\hat{k}$$

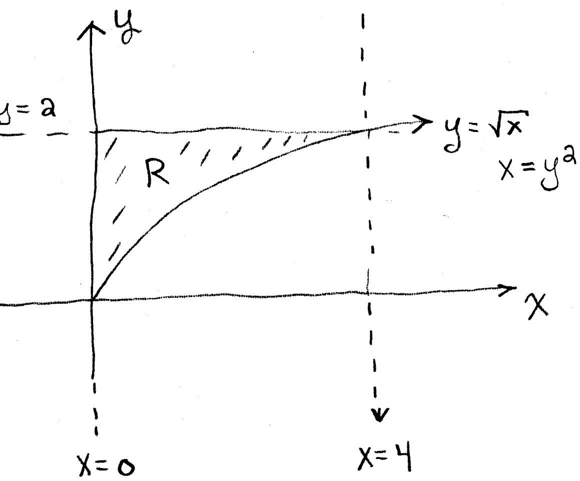
$$\vec{N} = \vec{\nabla} F(1,-3,2) = 12\hat{\imath} - 6\hat{\jmath} - 4\hat{k}$$

$$12(x-1) - 6(y+3) - 4(z-2) = 0$$

$$\boxed{12x - 6y - 4z = 22}$$

7. Sketch the region of integration, reverse the order of integration, and evaluate the new iterated integral by hand.

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{3}{2+y^2}\,dy\,dx$$



$$\int_0^2 \int_0^{y^2} \frac{3}{2+y^2}\,dx\,dy$$

$$= \int_0^2 \frac{3y^2}{2+y^2}\,dy = \int_0^2 \frac{3y^2+6}{2+y^2}\,dy - \int_0^2 \frac{6}{2+y^2}\,dy$$

$$= \int_0^2 3\,dy - \int_0^2 \frac{6}{2+y^2}\,dy = 3y - \frac{6}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}}\Big|_0^2 = \boxed{6 - \frac{6}{\sqrt{2}}\tan^{-1}\sqrt{2}}$$

8. Use Lagrange multipliers to find the extreme values of $f(x,y) = x^2y + 2y$ subject to $x^2 + y^2 = 25$.

$2xy = 2\lambda x \longrightarrow 2xy - 2\lambda x = 2x(y-\lambda) = 0$

$x^2 + 2 = 2\lambda y$

$x^2 + y^2 = 25$

$x = 0 \downarrow y = \pm 5$

$y = \lambda \downarrow$

$x^2 + 2 = 2y^2$

$2y^2 - 2 + y^2 = 25$

$3y^2 = 27$

$y = \pm 3$

$x = \pm 4$

CRIT POINTS ARE

$(0,5), (0,-5), (4,3),$

$(-4,3), (4,-3), (-4,-3)$

MAX IS $f(4,3) = f(-4,3) = 54$

MIN IS $f(4,-3) = f(-4,-3) = -54$

9. Let $\vec{r}(t) = 2t\hat{\imath} + t^2\hat{\jmath} + 2t\hat{k}$. Find the principal unit normal vector at $t = 0$.

$$\vec{r}\,'(t) = 2\hat{\imath} + 2t\hat{\jmath} + 2\hat{k} = 2(\hat{\imath} + t\hat{\jmath} + \hat{k})$$

$$|\vec{r}\,'(t)| = 2\sqrt{2 + t^2}$$

$$\hat{T}(t) = \frac{1}{\sqrt{2+t^2}}(\hat{\imath} + t\hat{\jmath} + \hat{k}) \qquad \hat{T}\,'(t) = \frac{\sqrt{2+t^2}\,(\hat{\jmath}) - (\hat{\imath}+t\hat{\jmath}+\hat{k})\left(\frac{1}{2}\right)(2+t^2)^{-1/2}(2t)}{2+t^2}$$

$$\hat{T}\,'(0) = \frac{\sqrt{2}\,\hat{\jmath}}{2}$$

$$\hat{N}(0) = \frac{\hat{T}\,'(0)}{|\hat{T}\,'(0)|} = \hat{\jmath}$$

10. Show that $\vec{F}(x, y) = \frac{1}{2}xy\hat{\imath} + \frac{1}{4}x^2\hat{\jmath}$ is a conservative vector field. Then use any method to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is any smooth curve from $(0, 0)$ to $(1, 1)$. Evaluate your integral by hand.

$$\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath} = \frac{1}{2}xy\,\hat{\imath} + \frac{1}{4}x^2\,\hat{\jmath}$$

$$\frac{\partial P}{\partial y} = \frac{1}{2}x = \frac{\partial Q}{\partial x} = \frac{1}{2}x \Rightarrow \vec{F} \text{ IS CONSERVATIVE!}$$

LET C BE THE LINE SEGMENT $\underbrace{y = x}_{dy = dx}$, $0 \le x \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \frac{1}{2}x^2\,dx + \frac{1}{4}x^2\,dx = \int_0^1 \frac{3}{4}x^2\,dx = \boxed{\frac{1}{4}}$$

11. The temperature distribution inside a containment vessel is given by

$$T(x,y,z) = \frac{4xe^{y^2}z^3}{x^2+1},$$

where the x, y, z are measured in centimeters and T is measured in degrees Celsius. Find a unit vector in the direction of maximum temperature increase at the point $(1, 0, 1)$.

$$\vec{\nabla} T(x,y,z) = \left(\frac{(x^2+1)(4e^{y^2}z^3) - (2x)(4xe^{y^2}z^3)}{(x^2+1)^2} \right) \hat{i}$$

$$+ \frac{8xye^{y^2}z^3}{(x^2+1)} \hat{j} + \frac{12xe^{y^2}z^2}{(x^2+1)} \hat{k}$$

$$\vec{\nabla} T(1,0,1) = 0\hat{i} + 0\hat{j} + 6\hat{k}$$

$$\frac{1}{6} \vec{\nabla} T(1,0,1) = \boxed{\hat{k}}$$

12. Let $\vec{r}(t) = 2\sqrt{t}\hat{\imath} + \frac{5\sin t}{t}\hat{\jmath} + (t^2+2)\hat{k}$.

(a) What is the domain of $\vec{r}$?

$\{t : t > 0\}$

(b) Find $\lim_{t \to 0} \vec{r}(t)$.

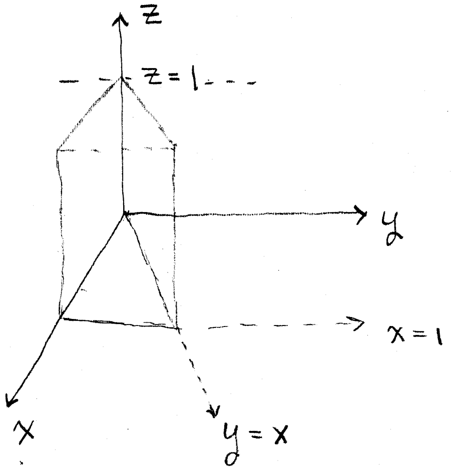
$$\lim_{t\to 0} 2\sqrt{t}\,\hat{i} + \lim_{t \to 0} \frac{5\sin t}{t}\hat{j} + \lim_{t\to 0}(t^2+2)\hat{k}$$

$$= \boxed{5\hat{j} + 2\hat{k}}$$

(c) What is the formal name we gave to the unit vector in the direction of $d\vec{r}/dt$?

UNIT TANGENT VECTOR

13. A triangular prism in the 1st octant is bounded by the planes $y = 0$, $y = x$, $x = 1$, $z = 0$, and $z = 1$. The density of the prism at the point (x, y, z) is given by $\rho(x, y, z) = x^3 + y^2 + z + 1$. Find the z-coordinate of the center of mass. Use your calculator or CAS to evaluate the required integrals.

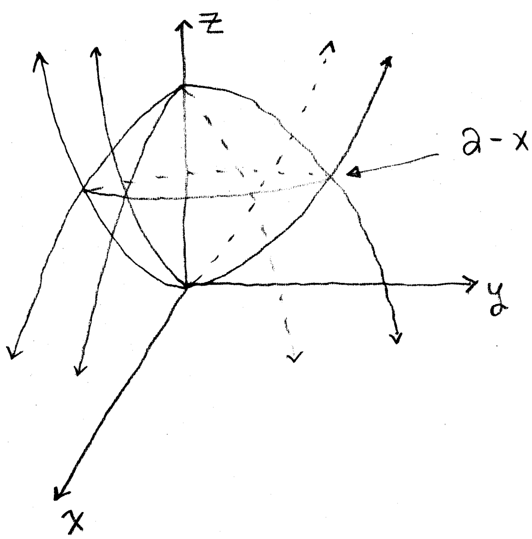


$$M = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \int_{z=0}^{z=1} (x^3 + y^2 + z + 1)\, dz\, dy\, dx = 31/30$$

$$M_{xy} = \int_0^1 \int_0^x \int_0^1 z(x^3 + y^2 + z + 1)\, dz\, dy\, dx = 67/120$$

$$\boxed{\bar{z} = \frac{M_{xy}}{M} = \frac{67}{124} \approx 0.54}$$

14. Use cylindrical coordinates to find the volume of the solid bounded by the paraboloids $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$. Use your calculator or CAS to evaluate the required integral.

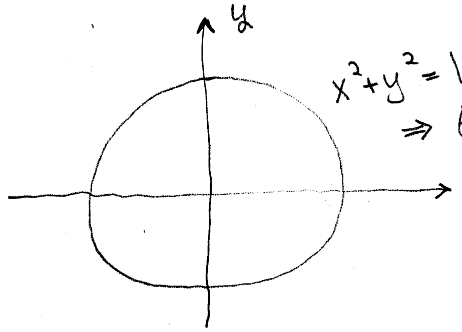


$2 - x^2 - y^2 = x^2 + y^2$

$x^2 + y^2 = 1$

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r^2}^{z=2-r^2} r\, dz\, dr\, d\theta$$

$$= \boxed{\pi}$$



$x^2 + y^2 = 1$

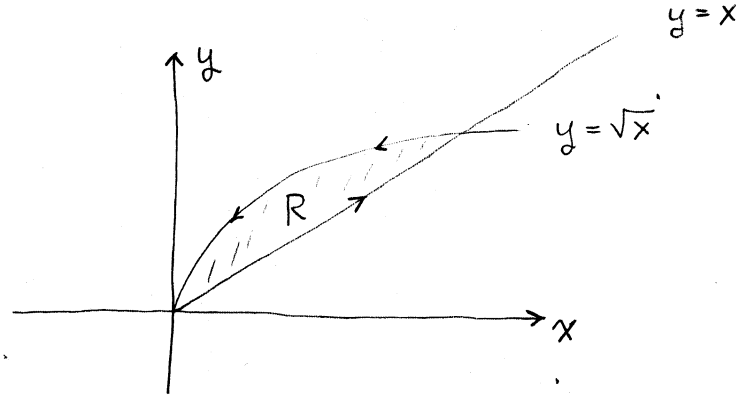
$\Rightarrow \theta = 0$ to 2π

$r = 0$ to 1

15. Use Green's Theorem to evaluate

$$\int_C \cos y \, dx + (xy - x \sin y) \, dy,$$

where C is the positively-oriented boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$. Evaluate your integral by hand.



$$\iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{y - \sin y - (-\sin y) = y} dA$$

$$\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{x}} y \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \left[(\sqrt{x})^2 - x^2\right] dx$$

$$= \frac{1}{2} \int_0^1 (x - x^2) \, dx = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \boxed{\frac{1}{12}}$$