Math 173 - Test 1

February 21, 2013

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (7 points) The vector $\vec{u}$ lies in the xy-plane, has magnitude 7, and makes an angle of $150°$ with the positive x-axis. The vector $\vec{w}$ is given by $\vec{w} = -6\hat{\imath} + \hat{\jmath} + 4\hat{k}$. Determine the magnitude of $3\vec{w} - 2\vec{u}$.

$$\vec{u} = 7\cos 150° \hat{\imath} + 7\sin 150° \hat{\jmath} = -\frac{7\sqrt{3}}{2}\hat{\imath} + \frac{7}{2}\hat{\jmath}$$

$$\vec{w} = -6\hat{\imath} + \hat{\jmath} + 4\hat{k}$$

$$3\vec{w} - 2\vec{u} = (-18 + 7\sqrt{3})\hat{\imath} + (-4)\hat{\jmath} + 12\hat{k}$$

$$\|3\vec{w} - 2\vec{u}\| = \sqrt{(-18+7\sqrt{3})^2 + (-4)^2 + (12)^2} \approx \underline{13.947}$$

2. (6 points) Use vectors to determine whether the points are collinear. Explain your reasoning.

$P(3, -7, 2) \qquad Q(-8, -9, 5) \qquad R(-30, -13, 8)$

$$\vec{PQ} = -11\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$$

$$\vec{PR} = -33\hat{\imath} - 6\hat{\jmath} + 6\hat{k}$$

THE POINTS ARE NOT COLLINEAR BECAUSE $\vec{PQ}$ IS NOT A SCALAR MULTIPLE OF $\vec{PR}$.

3. (6 points) Find a vector of length 5 that is orthogonal to $-3\hat{\imath} + 2\hat{\jmath}$.

$$\left.\begin{aligned} \vec{u} &= -3\hat{\imath} + 2\hat{\jmath} \\ \vec{w} &= 2\hat{\imath} + 3\hat{\jmath} \end{aligned}\right\} \quad \vec{u}\cdot\vec{w} = 0 \Rightarrow \vec{u} \text{ AND } \vec{w} \text{ ARE ORTHOGONAL.}$$

$$\frac{5}{\|\vec{w}\|}\vec{w} = \frac{5}{\sqrt{13}}(2\hat{\imath} + 3\hat{\jmath}) = \boxed{\frac{10}{\sqrt{13}}\hat{\imath} + \frac{15}{\sqrt{13}}\hat{\jmath}}$$

4. (2 points) If you were given two nonparallel vectors, how could you find a nonzero vector orthogonal to both?

DETERMINE THE CROSS PRODUCT.

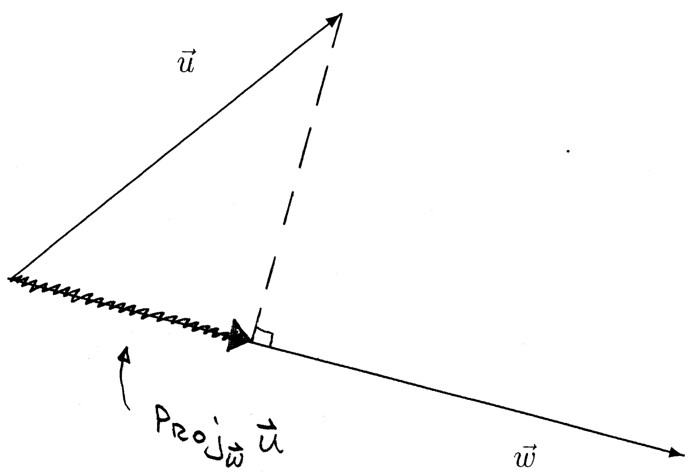
THE CROSS PRODUCT OF TWO VECTORS IS ORTHOG TO BOTH.

5. (6 points) Find the angle between $\vec{u} = 5\hat{\imath} + 7\hat{\jmath} - \hat{k}$ and $\vec{v} = \hat{\imath} + 3\hat{k}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\ \|\vec{v}\|} = \frac{5-3}{\sqrt{75}\sqrt{10}} = \frac{2}{\sqrt{750}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{750}}\right) \approx \underline{85.81^\circ}$$

6. (8 points) The vectors $\vec{u}$ and $\vec{w}$ are shown below. (a) Sketch $\text{proj}_{\vec{w}}\,\vec{u}$.



(b) Suppose $\vec{u} = 2\hat{\imath} + 3\hat{k}$ and $\vec{w} = \hat{\imath} + 5\hat{\jmath} - 2\hat{k}$. Compute $\text{proj}_{\vec{w}}\,\vec{u}$.

$$\text{proj}_{\vec{w}}\,\vec{u} = \frac{\vec{u}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\,\vec{w} = \frac{2-6}{1+25+4}\,\vec{w} = \frac{-4}{30}\,\vec{w}$$

$$= \boxed{-\frac{2}{15}\left(\hat{\imath} + 5\hat{\jmath} - 2\hat{k}\right)}$$

7. (12 points) The points $P(1,0,5)$, $Q(2,2,-3)$, and $R(-3,8,1)$ are the vertices of a triangle.

(a) Find the area of the triangle.

$\vec{PQ} = \hat{i} + 2\hat{j} - 8\hat{k}$

$\vec{PR} = -4\hat{i} + 8\hat{j} - 4\hat{k}$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -8 \\ -4 & 8 & -4 \end{vmatrix}$$

$$= \hat{i}\,(56) - \hat{j}\,(-36) + \hat{k}\,(16)$$

AREA OF $\Delta PQR = \frac{1}{2} \| \vec{PQ} \times \vec{PR} \|$

$$= \frac{1}{2}\sqrt{56^2 + 36^2 + 16^2}$$

$$= \frac{1}{2}\sqrt{4688} = \sqrt{1172}$$

$$\approx \underline{34.23}$$

(b) Find an equation of the plane containing the triangle.

Using $\vec{N} = 56\hat{i} + 36\hat{j} + 16\hat{k}$

AND $P(1,0,5)$ ...

$$56(x-1) + 36(y-0) + 16(z-5) = 0$$

$$\boxed{56x + 36y + 16z = 136}$$

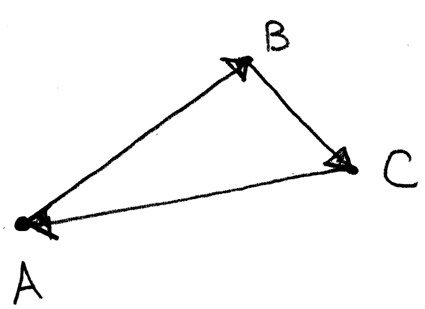
(c) Find a set of parametric equations for the line segment $\overline{PQ}$.

Using $\vec{PQ} = \hat{i} + 2\hat{j} - 8\hat{k}$

AND $P(1,0,5)$ ...

$$\boxed{\begin{aligned} x &= 1 + t \\ y &= 2t \qquad 0 \le t \le 1 \\ z &= 5 - 8t \end{aligned}}$$

8. (4 points) Let A, B, and C be the vertices of a triangle. Determine $\vec{AB} + \vec{BC} + \vec{CA}$.



$$\underbrace{\vec{AB} + \vec{BC}} + \vec{CA} = \vec{0}$$

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\vec{AC} + \vec{CA} = \vec{0}$$

9. (10 points) An object is launched from the ground at 30 m/s. Find the initial angle if the object will reach its maximum height 38 m downrange. Use $g = 9.8\,\text{m/s}^2$. (Hint: You may need to use the identity $2\sin\theta\cos\theta = \sin 2\theta$.)

$$\vec{r}(t) = 30\cos\theta\, t\,\hat{i} + \left(-4.9t^2 + 30\sin\theta\, t\right)\hat{j}$$

$$30\cos\theta\, t = 38$$

$$-9.8t + 30\sin\theta = 0 \Rightarrow t = \frac{30\sin\theta}{9.8} \Rightarrow 30\cos\theta\left(\frac{30\sin\theta}{9.8}\right) = 38$$

$$\Rightarrow 900\cos\theta\sin\theta = 372.4$$

$$\Rightarrow 2\cos\theta\sin\theta = \frac{372.4}{450} \Rightarrow \sin 2\theta = \frac{372.4}{450} \Rightarrow \underline{\theta = 27.924^\circ}$$

10. (4 points) Find a vector-valued function whose graph is the line with symmetric equations:

$$\frac{x-1}{3} = \frac{y+2}{5} = \frac{z-3}{7} \Rightarrow \begin{aligned} x &= 1+3t \\ y &= -2+5t \\ z &= 3+7t \end{aligned}$$

$$\boxed{\vec{r}(t) = (1+3t)\hat{i} + (-2+5t)\hat{j} + (3+7t)\hat{k}}$$

11. (6 points) Consider the plane $5x + 2y - z = 7$.

(a) Find a vector normal to the plane.

$$\vec{N} = 5\hat{i} + 2\hat{j} - \hat{k}$$

(b) Find two points in the plane.

$\overset{P}{(1,1,0)}$ AND $\overset{Q}{(0,0,-7)}$ BOTH SATISFY THE EQUATION!

(c) Connect your points to form a vector. Then show your vector is orthogonal to the normal vector.

$$\vec{PQ} = -\hat{i} - \hat{j} - 7\hat{k}$$

$$\vec{PQ}\cdot\vec{N} = 5(-1) + 2(-1) + (-1)(-7) = 0$$

12. (3 points) Suppose $\vec{r}(t)$ describes a line in space. What can be said about $\hat{T}'(t)$?

IF $\vec{r}(t)$ DESCRIBES A LINE, $\vec{r}\,'(t)$ AND $\hat{T}(t)$ ARE CONSTANT.

THEREFORE, $\hat{T}'(t) = \vec{0}$.

13. (6 points) Let $\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + \ln t\hat{k}$. Compute $\hat{T}(1)$.

$$\vec{r}\,'(t) = \hat{\imath} + 2t\hat{\jmath} + \frac{1}{t}\hat{k}$$

$$\vec{r}\,'(1) = \hat{\imath} + 2\hat{\jmath} + \hat{k}$$

$$\hat{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \frac{\hat{\imath} + 2\hat{\jmath} + \hat{k}}{\sqrt{6}}$$

14. (10 points) A projectile is falling in such a way that its motion is described by

$$\vec{r}(t) = (25\sqrt{3}t)\hat{\imath} + (-16t^2 + 25t + 25)\hat{\jmath}.$$

(a) When does the projectile reach is maximum height?

$$-32t + 25 = 0$$

$$\Rightarrow \quad t = \frac{25}{32} = 0.78125$$

(b) Set up, but do not evaluate, the definite integral that gives the length of the path of the projectile from $t = 0$ until it reaches its max height.

$$\int_0^{25/32} \|\vec{r}\,'(t)\|\,dt = \int_0^{25/32} \sqrt{\left[25\sqrt{3}\right]^2 + \left[-32t+25\right]^2}\,dt$$

$$\approx 35.624$$

15. (2 points) Describe the motion of a particle if its normal component of acceleration is 0.

THE DIRECTION IS NOT CHANGING.

THE PARTICLE IS MOVING ALONG A STRAIGHT LINE.

16. (2 points) Describe the motion of a particle if its tangential component of acceleration is 0.

THE SPEED OF THE PARTICLE IS NOT CHANGING. ONLY THE DIRECTION IS CHANGING.

17. (6 points) Suppose a particle moves along the curve from left to right. Sketch and label each of the following:

(a) the unit tangent vector at the point of greatest curvature

(b) a point where the principal unit normal vector does not exist

(c) the principal unit normal vector at the point P