

**Math 173 - Test 2a**  
March 21, 2013

Name key \_\_\_\_\_  
Score \_\_\_\_\_

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Find a set of parametric equations for the line normal to the surface at the point  $(-1, 1, -1)$ .

$$x^2y^3 - y^2z + 2xz^3 = 4$$

$$F(x, y, z) = x^2y^3 - y^2z + 2xz^3$$

$$\vec{\nabla} F(x, y, z) = (2xy^3 + 2z^3)\hat{i} + (3x^2y^2 - 2yz)\hat{j} + (-y^2 + 6xz^2)\hat{k}$$

$$\vec{\nabla} F(-1, 1, -1) = -4\hat{i} + 5\hat{j} - 7\hat{k}$$

DIRECTION :  $-4\hat{i} + 5\hat{j} - 7\hat{k}$

POINT :  $(-1, 1, -1)$   $\Rightarrow$

$$\boxed{\begin{aligned} x &= -4t - 1 \\ y &= 5t + 1 \\ z &= -7t - 1 \end{aligned}}$$

2. (6 points) Show that the limit does not exist:

$$\lim_{(x,y) \rightarrow (2,1)} \frac{2y^2 - x}{x^2 - 4y}$$

Two PATH TEST...

$$\text{Along } x=2 : \lim_{y \rightarrow 1} \frac{2y^2 - 2}{4 - 4y} = \lim_{y \rightarrow 1} \frac{2(y-1)(y+1)}{4(1-y)} = \lim_{y \rightarrow 1} \frac{-2(y+1)}{-4} = \underline{-1}$$

$$\text{Along } y=1 : \lim_{x \rightarrow 2} \frac{2-x}{x^2-4} = \lim_{x \rightarrow 2} \frac{(2-x)}{(x+2)(x-2)} = \lim_{x \rightarrow 2} \frac{-1}{x+2} = \underline{-\frac{1}{4}}$$

Two DIFFERENT LIMITS  $\Rightarrow$

3. (8 points) Use differentials to approximate  $\Delta z$  as  $(x, y)$  changes from  $(1, 1)$  to  $(1.01, 0.98)$ .

$$f(x, y) = \sqrt{5xy + 4x^2y^4}$$

$$\Delta z \approx \frac{1}{2}(5xy + 4x^2y^4)^{-1/2} (5y + 8x^2y^3) \Delta x + \frac{1}{2}(5xy + 4x^2y^4)^{-1/2} (5x + 16x^2y^3) \Delta y$$

$$x = 1, y = 1, \Delta x = 0.01, \Delta y = -0.02$$

$$\Delta z \approx \frac{1}{2}(9)^{-1/2} (13)(0.01) + \frac{1}{2}(9)^{-1/2} (21)(-0.02)$$

$$\Delta z \approx \frac{1}{6}(0.13) - \frac{1}{6}(0.42) = \boxed{-\frac{0.29}{6} \approx -0.0483}$$

4. (5 points) Suppose  $w = f(x, y)$  is a differentiable function of  $x$  and  $y$ . Let  $x = u - v$  and  $y = v - u$ . Use the chain rule to determine  $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$ .

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial w}{\partial x}(1) + \frac{\partial w}{\partial y}(-1)$$

$$\Rightarrow \boxed{\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial w}{\partial x}(-1) + \frac{\partial w}{\partial y}(1)$$

5. (6 points) Assume that the following equation implicitly defines  $z$  as a function of  $x$  and  $y$ . Find  $\partial z / \partial x$  at the point  $(\frac{\pi}{4}, 0, \pi)$ .

$$\underbrace{\tan(x+y) + \sin(y+z)}_{F(x,y,z)} = 1$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\sec^2(x+y)}{\cos(y+z)}$$

$$\left. \frac{\partial z}{\partial x} \right|_{(\frac{\pi}{4}, 0, \pi)} = \frac{-\sec^2(\frac{\pi}{4})}{\cos(\pi)} = \frac{-(\sqrt{2})^2}{-1} = \boxed{2}$$

6. (10 points) Find and classify the critical points of  $f(x, y) = 2x^2 + 2xy + y^2 - 2x - 3$ .

$$f_x(x, y) = 4x + 2y - 2 = 0$$

$$f_y(x, y) = 2x + 2y = 0 \Rightarrow x = -y \Rightarrow 4(-y) + 2y - 2 = 0 \Rightarrow y = -1 \\ x = 1$$

Only crit pt is  $(1, -1)$

$$f_{xx}(x, y) = 4$$

$$f_{xy}(x, y) = 2 \quad d(1, -1) = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4 > 0$$

$$f_{yy}(x, y) = 2$$

$$d(1, -1) > 0 \text{ AND } f_{xx}(1, -1) > 0 \Rightarrow$$

7. (10 points) Consider the function  $g(x, y) = \tan^{-1}(\frac{y}{x})$ .

(a) Find an equation of the plane tangent to the graph of  $g$  at the point  $(1, 1, \frac{\pi}{4})$ .

$$F(x, y, z) = z - \tan^{-1}\left(\frac{y}{x}\right)$$

$$\nabla F(x, y, z) = \frac{y/x^2}{1+(\frac{y}{x})^2} \hat{i} - \frac{1}{1+(\frac{y}{x})^2} \hat{j} + \hat{k}$$

$$\nabla F(1, 1, \frac{\pi}{4}) = \frac{1}{2} \hat{i} - \frac{1}{2} \hat{j} + \hat{k}$$

= NORMAL VECTOR

PLANE:

$$\frac{1}{2}(x-1) - \frac{1}{2}(y-1) + \left(z - \frac{\pi}{4}\right) = 0$$

(b) Find the linearization of  $g$  at  $(1, 1)$ .

From above, solving for  $z$ , ...

$$L(x, y) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

(c) Use your linearization to approximate  $g(1.05, 0.97)$ .

$$g(1.05, 0.97) \approx L(1.05, 0.97) = \frac{\pi}{4} - \frac{1}{2}(0.05) + \frac{1}{2}(-0.03)$$

$$\approx \underline{\underline{0.7454}}$$

Actual value:  $g(1.05, 0.97) = 0.7458148895\dots$

8. (4 points) Suppose  $z = 2xe^{5y} - 3ye^{-x}$ . Find  $\frac{\partial^2 z}{\partial x \partial y}$ .

$$\frac{\partial z}{\partial y} = 10xe^{5y} - 3e^{-x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 10e^{5y} + 3e^{-x}$$

9. (10 points) Consider the surface described by the equation  $4x^2 - 3y^2 + 12z^2 + 3 = 0$ .

(a) Describe the level curve at  $y = 2$ .

$$4x^2 + 12z^2 = 12 - 3$$

$$4x^2 + 12z^2 = 9$$

$$\frac{x^2}{9/4} + \frac{z^2}{9/12} = 1$$

ELLIPSE CENTERED AT  
(0,0).

(b) Is there a level curve at  $y = 0$ ? Explain.

$$4x^2 + 12z^2 = -3$$

No way! No real  $(x,z)$  make THIS TRUE!

(c) Describe the level curve  $z = 1$ .

$$4x^2 - 3y^2 = -15$$

$$3y^2 - 4x^2 = 15$$

$$\frac{y^2}{15/3} - \frac{x^2}{15/4} = 1$$

HYPERBOLA CENTERED AT  
(0,0)

(d) What are the level curves in planes parallel to the  $yz$ -plane?

$$x = k$$

$$-3y^2 + 12z^2 = -3 - 4k^2$$

THIS EQUATION DESCRIBES A

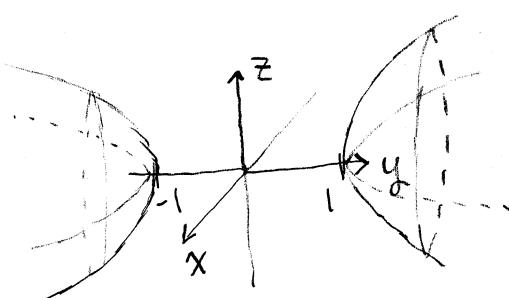
(e) Identify the quadric surface and roughly sketch the graph.

HYPERBOLA.

THE SURFACE IS A HYPERBOLOID OF TWO SHEETS,

OPENING UP THE Y-AXIS, WITH VERTICES AT  $(0, 1, 0)$  AND

$(0, -1, 0)$



10. (5 points) Determine the limit:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - 4y^4}{x^2 + 2y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - 2y^2)(x^2 + 2y^2)}{(x^2 + 2y^2)}$$

$$= \lim_{(x,y) \rightarrow (0,0)} (x^2 - 2y^2) = \boxed{0}$$

11. (8 points) Consider the function  $f(x, y, z) = xy^3 - \frac{\sqrt{x}}{z^2}$ .

(a) Find the directional derivative of  $f$  at  $(4, 1, 2)$  in the direction of  $\hat{i} + 2\hat{j} - \hat{k}$ .

$$\vec{\nabla} f(x, y, z) = \left(y^3 - \frac{\frac{1}{2}x^{-\frac{1}{2}}}{z^2}\right)\hat{i} + (3xy^2)\hat{j} + \frac{2\sqrt{x}}{z^3}\hat{k}$$

$$\vec{\nabla} f(4, 1, 2) = \left(1 - \frac{1}{16}\right)\hat{i} + 12\hat{j} + \frac{1}{2}\hat{k} = \frac{15}{16}\hat{i} + 12\hat{j} + \frac{1}{2}\hat{k}$$

$$\|\hat{i} + 2\hat{j} - \hat{k}\| = \sqrt{6}$$

DIRECTIONAL DERIVATIVE IS

$$\boxed{\frac{1}{\sqrt{6}} \left(\frac{15}{16} + 24 - \frac{1}{2}\right) \approx 9.977}$$

(b) Find a unit vector in the direction of maximum increase of  $f$ .

Assuming AT  $(4, 1, 2)$ , we HAVE MAX INCREASE

IN DIRECTION OF  $\vec{\nabla} f(4, 1, 2)$ :

$$\boxed{\frac{\vec{\nabla} f(4, 1, 2)}{\|\vec{\nabla} f(4, 1, 2)\|} = \frac{1}{\sqrt{\left(\frac{15}{16}\right)^2 + (12)^2 + \left(\frac{1}{2}\right)^2}} \left(\frac{15}{16}\hat{i} + 12\hat{j} + \frac{1}{2}\hat{k}\right)}$$

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due Monday, April 1. You must work individually on this test.

1. (6 points) Use the definition of differentiability to show that  $f(x, y) = x + x^2y - 2y$  is differentiable everywhere.

$$f_x(x, y) = 1 + 2xy$$

$$f_y(x, y) = x^2 - 2$$

$$\begin{aligned}
 \Delta z &= f(x+\Delta x, y+\Delta y) - f(x, y) \\
 &= [x + \Delta x + (x + \Delta x)^2(y + \Delta y) - 2(y + \Delta y)] - [x + x^2y - 2y] \\
 &= [x + \Delta x + x^2y + x^2\Delta y + 2xy\Delta x + 2x\Delta x\Delta y + y\Delta x^2 + \Delta x^2\Delta y \\
 &\quad - 2y - 2\Delta y] - [x + x^2y - 2y] \\
 &= \underline{\Delta x} + \underline{x^2\Delta y} + \underline{2xy\Delta x} + \underline{2x\Delta x\Delta y} + \underline{y\Delta x^2} + \underline{\Delta x^2\Delta y} - \underline{2\Delta y} \\
 &= (1 + 2xy)\Delta x + (x^2 - 2)\Delta y + (2x\Delta y + y\Delta x)\Delta x + (\Delta x^2)\Delta y \\
 &= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y
 \end{aligned}$$

WHERE  $\epsilon_1 = 2x\Delta y + y\Delta x$  AND  $\epsilon_2 = \Delta x^2$

SINCE THIS IS TRUE FOR ALL  $(x, y)$  AND  $\epsilon_1 \rightarrow 0$  AND  $\epsilon_2 \rightarrow 0$   
 AS  $(\Delta x, \Delta y) \rightarrow (0, 0)$ ,  $f$  IS DIFFERENTIABLE FOR ALL  $(x, y)$ .

2. (7 points) Find and classify the critical points of  $g(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$ .

$$g_x(x, y) = -6xy - 6x = 0 \Rightarrow -6x(y+1) = 0 \Rightarrow x=0 \text{ or } y=-1$$

$$g_y(x, y) = 3y^2 - 3x^2 - 6y = 0 \quad x=0 \Rightarrow \quad y=-1 \Rightarrow \\ 3y^2 - 6y = 0 \quad 3+6 = 3x^2 \\ 3y(y-2) = 0 \quad 3 = x^2 \\ y=0, y=2 \quad x = \pm\sqrt{3}$$

$$g_{xx}(x, y) = -6y - 6$$

$$3y(y-2) = 0 \quad 3 = x^2 \\ y=0, y=2 \quad x = \pm\sqrt{3}$$

$$g_{yy}(x, y) = 6y - 6$$

$$g_{xy}(x, y) = -6x$$

Crit points:

$$(0, 0), (0, 2), (\sqrt{3}, -1), (-\sqrt{3}, -1)$$

$$d(x, y) = \begin{vmatrix} -6y-6 & -6x \\ -6x & 6y-6 \end{vmatrix}$$

$$= -36y^2 + 36 - 36x^2$$

$$d(0, 0) = 36, g_{xx}(0, 0) = -6 \Rightarrow g(0, 0) = 1 \text{ IS A RELATIVE MAX}$$

$$d(0, 2) = -108 \Rightarrow (0, 2, g(0, 2)) = (0, 2, -3) \text{ IS A SADDLE PT}$$

$$d(\sqrt{3}, -1) = -108 \Rightarrow (\sqrt{3}, -1, g(\sqrt{3}, -1)) = (\sqrt{3}, -1, -3) \text{ IS A SADDLE PT}$$

$$d(-\sqrt{3}, -1) = -108 \Rightarrow (-\sqrt{3}, -1, g(-\sqrt{3}, -1)) = (-\sqrt{3}, -1, -3) \text{ IS A SADDLE PT.}$$

3. (4 points) The period  $T$  of a pendulum of length  $L$  is  $T = 2\pi\sqrt{L/g}$ , where  $g$  is the acceleration due to gravity. A pendulum is moved from a place where  $g = 32.09 \text{ ft/s}^2$  to a place where  $g = 32.23 \text{ ft/s}^2$ . Also accompanying the change in  $g$  is a change in temperature that results in the length of the pendulum decreasing from 2.50 ft to 2.48 ft. Use differentials to approximate the corresponding change in the period of the pendulum.

$$\Delta T \approx \frac{\partial T}{\partial L} \Delta L + \frac{\partial T}{\partial g} \Delta g$$

$$\frac{\partial T}{\partial L} = \pi \left( \frac{L}{g} \right)^{-\frac{1}{2}} \left( \frac{1}{g} \right) \quad ; \quad \text{when } g = 32.09 \text{ and } L = 2.50,$$

$$\frac{\partial T}{\partial g} = \pi \left( \frac{L}{g} \right)^{-1/2} \left( -\frac{L}{g^2} \right) \quad \left. \begin{array}{l} \frac{\partial T}{\partial L} \approx 0.35075 \\ \frac{\partial T}{\partial g} \approx -0.027325 \end{array} \right\}$$

$$\Delta T \approx (0.35075)(-0.02) + (-0.027325)(0.14)$$

$$\approx -0.01084 \text{ sec}$$

4. (3 points) Suppose  $g(x, y) = \frac{4x^2y^2}{x^2 + y^2}$  was to be defined at  $(0, 0)$  in such a way that it is continuous everywhere. What value must it be given at  $(0, 0)$ ?

$$\lim_{\substack{(x,y) \rightarrow (0,0)}} \frac{4x^2y^2}{x^2+y^2} = \lim_{r \rightarrow 0} \frac{4r^4 \sin^2 \theta \cos^2 \theta}{r^2} = \lim_{r \rightarrow 0} 4r^2 \sin^2 \theta \cos^2 \theta$$

$\brace{}$

POLAR COORDS

$$= 4 \sin^2 \theta \cos^2 \theta \lim_{r \rightarrow 0} r^2$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

=