

Show all work. Supply explanations when necessary.

1. (5 points) Let $\vec{r}(t) = 2 \cos t \hat{i} + 2 \sin t \hat{j} + 5 \hat{k}$. Show that $\vec{r}(t)$ and $\vec{r}'(t)$ are orthogonal.

$$\vec{r}'(t) = -2 \sin t \hat{i} + 2 \cos t \hat{j}$$

$$\vec{r}(t) \cdot \vec{r}'(t) = -4 \cos t \sin t + 4 \sin t \cos t + 5(0)$$

$$= 0$$

$\Rightarrow \vec{r}(t)$ AND $\vec{r}'(t)$ ARE ORTHOG.

2. (5 points) Find $\vec{r}(t)$ if $\vec{r}'(t) = t^2 \hat{i} + 4t^3 \hat{j} - t^2 \hat{k}$ and $\vec{r}(0) = \hat{j}$.

$$\vec{r}(t) = \left(\frac{t^3}{3} + c_1 \right) \hat{i} + \left(t^4 + c_2 \right) \hat{j} + \left(-\frac{t^3}{3} + c_3 \right) \hat{k}$$

$$\vec{r}(0) = \hat{j} \Rightarrow c_1 = c_3 = 0 \text{ AND } c_2 = 1$$

$$\boxed{\vec{r}(t) = \frac{t^3}{3} \hat{i} + (t^4 + 1) \hat{j} - \frac{t^3}{3} \hat{k}}$$

3. (2 points) If a car's speedometer is constant, which component of the acceleration is zero, the tangential component or the normal component? Explain.

IF THE SPEEDOMETER IS CONSTANT, THE SPEED IS NOT
 CHANGING, BUT THE DIRECTION MIGHT BE. THE ACCELERATION
 COMPONENT THAT ACTS TO CHANGE THE SPEED MUST BE ZERO
 \Rightarrow TANGENTIAL COMP, a_T , IS ZERO.

4. (5 points) Set up the integral that gives the length of the arc of the twisted cubic $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ from the origin to the point (2, 4, 8). Use the numerical integration feature on your calculator to approximate the arc length.

$$\vec{r}'(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\vec{r}(2) = 2\hat{i} + 4\hat{j} + 8\hat{k}$$

$$\vec{r}(0) = \vec{0}$$

$$\text{Arc Length} = \int_0^2 \sqrt{1 + 4t^2 + 9t^4} dt \approx 9.57$$

5. (12 points) Determine each limit or explain why the limit does not exist.

$$(a) \lim_{(x,y) \rightarrow (2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \cdot \frac{\sqrt{x-y}+1}{\sqrt{x-y}+1} = \lim_{(x,y) \rightarrow (2,1)} \frac{(x-y-1)(\sqrt{x-y}+1)}{(x-y-1)} = \sqrt{2-1} + 1 = \boxed{2}$$

$$(b) \lim_{(x,y) \rightarrow (0,0)} \frac{y}{x^2+y}$$

Along $x=0$:

$$\lim_{y \rightarrow 0} \frac{y}{y} = \lim_{y \rightarrow 0} 1 = 1$$

Along $y=0$:

$$\lim_{x \rightarrow 0} \frac{0}{x^2} = \lim_{x \rightarrow 0} 0 = 0$$

LIMIT DNE
By TWO-PATH TEST.

$$(c) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{x^2+y^2}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\lim_{r \rightarrow 0} \frac{r^4 \sin^2 \theta \cos^2 \theta}{r^2} = \lim_{r \rightarrow 0} r^2 \sin^2 \theta \cos^2 \theta$$

$$= \boxed{0}$$

6. (12 points) A projectile is fired with an initial speed of 500 m/s and angle of elevation 30° . (Use $g = 9.8 \text{ m/s}^2$ for this problem.)

(a) Find the range of the projectile.

$$-4.9t^2 + 250t = 0$$

$$t(-4.9t + 250) = 0$$

$$t=0 \quad \text{or} \quad t = \frac{250}{4.9}$$

$$\vec{r}(t) = 500 \cos 30^\circ t \hat{i} + (-4.9t^2 + 500 \sin 30^\circ t) \hat{j}$$

$$\vec{r}(t) = 250\sqrt{3}t \hat{i} + (-4.9t^2 + 250t) \hat{j}$$

$$\text{Range} = 250\sqrt{3} \left(\frac{250}{4.9} \right) \approx 22,092.5 \text{ m}$$

(b) Find the projectile's maximum height.

$$\begin{aligned} -9.8t + 250 &= 0 \\ \Rightarrow t &= \frac{250}{9.8} \end{aligned}$$

$$\begin{aligned} \text{MAX HEIGHT IS } &-4.9 \left(\frac{250}{9.8} \right)^2 + 250 \left(\frac{250}{9.8} \right) \\ &\approx 3188.8 \text{ m} \end{aligned}$$

(c) Find the speed of the projectile when it hits the ground.

$$\vec{r}'(t) = 250\sqrt{3} \hat{i} + (-9.8t + 250) \hat{j}$$

$$\vec{r}'\left(\frac{250}{4.9}\right) = 250\sqrt{3} \hat{i} - 250 \hat{j}$$

$$\|\vec{r}'\left(\frac{250}{4.9}\right)\| = 500 \text{ m/s}$$

7. (5 points) Find a vector-valued function whose graph is the line segment from $(-2, 5, 3)$ to $(8, 4, 1)$.

Q

P

$$\vec{PQ} = 10\hat{i} - \hat{j} - 2\hat{k}$$

Point $(-2, 5, 3)$

$$x = 10t - 2$$

$$y = -t + 5 \quad 0 \leq t \leq 1$$

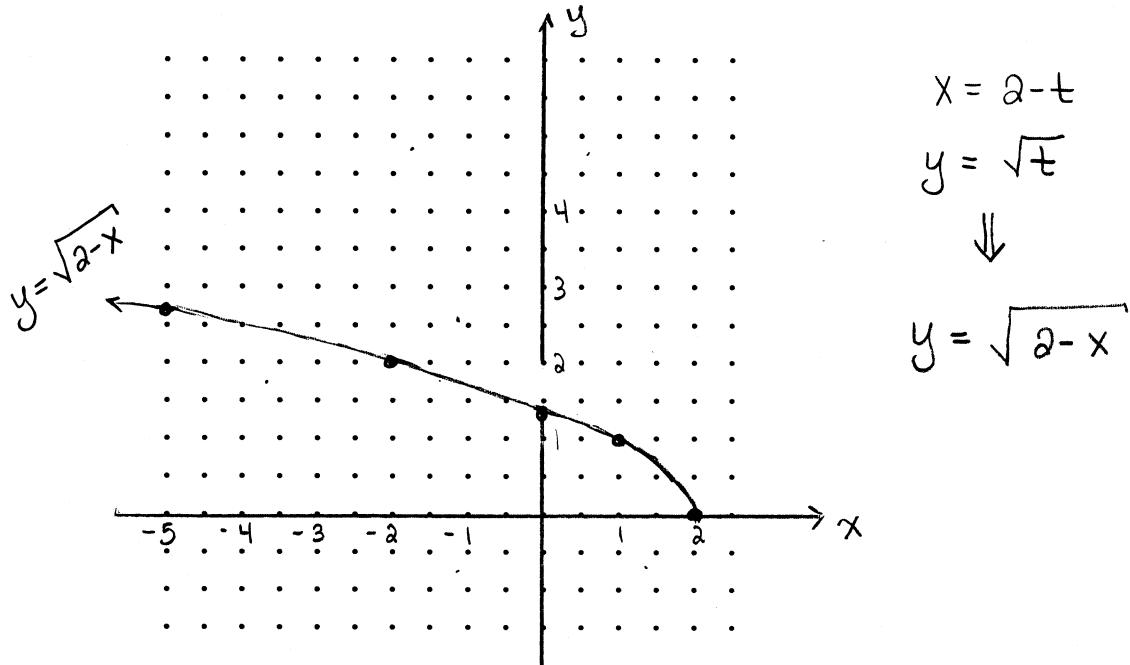
$$z = -2t + 3$$



$$\begin{aligned} \vec{r}(t) &= (10t - 2)\hat{i} + (-t + 5)\hat{j} \\ &\quad + (-2t + 3)\hat{k}, \end{aligned}$$

$$0 \leq t \leq 1$$

8. (5 points) Sketch the curve described by the vector-valued function $\vec{r}(t) = (2-t)\hat{i} + \sqrt{t}\hat{j}$.



9. (4 points) Show that for any twice-differentiable vector-valued function

$$\frac{d}{dt} [\vec{r}(t) \times \vec{r}'(t)] = \vec{r}(t) \times \vec{r}''(t).$$

EASY WAY...

$$\vec{u} \times \vec{u} = \vec{0} \text{ FOR ANY VECTOR } \vec{u}$$

$$\frac{d}{dt} [\vec{r} \times \vec{r}'] = (\vec{r} \times \vec{r}'') + (\cancel{\vec{r}' \times \vec{r}'}) = \vec{r} \times \vec{r}''.$$

HARD WAY...

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$\frac{d}{dt} [\vec{r} \times \vec{r}'] = \frac{d}{dt} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = \frac{d}{dt} [(yz' - y'z)\hat{i} - (xz' - x'z)\hat{j} + (xy' - x'y)\hat{k}]$$

= ...

10. (12 points) Consider the function $g(x, y) = \frac{8}{1+x^2+y^2}$.

(a) What is the domain of g ?

$$\text{Domain} = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R}^2$$

(b) What is the range of g ?

$$\text{Range} = \{z : 0 < z \leq 8\}$$

(c) Describe the level curve $g(x, y) = 1$.

$$1 = \frac{8}{1+x^2+y^2} \Rightarrow 1+x^2+y^2 = 8 \Rightarrow x^2+y^2 = 7.$$

CIRCLE CENTERED
AT $(0,0)$ WITH
RADIUS $\sqrt{7}$

(d) At which points is g continuous?

g IS CONTINUOUS EVERYWHERE.

(e) Compute the partial derivatives g_x and g_y .

$$g_x(x, y) = \frac{-16x}{(1+x^2+y^2)^2}$$

$$g_y(x, y) = \frac{-16y}{(1+x^2+y^2)^2}$$

$$g(x, y) = 8(1+x^2+y^2)^{-1}$$

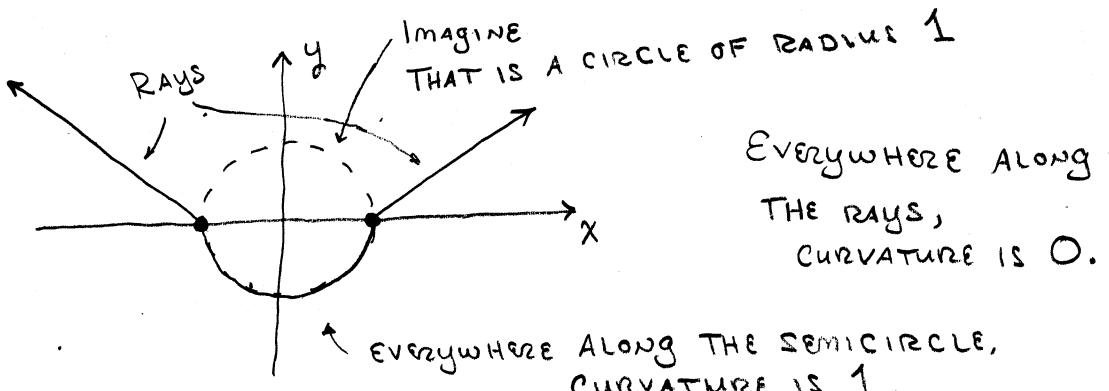
(f) Without actually computing the mixed partial derivatives g_{xy} and g_{yx} , would you expect them to be equal? Explain.

YES, g_{xy} AND g_{yx} WILL BE CONTINUOUS

EVERYWHERE. THEREFORE, ACCORDING TO

OUR THEOREM, THEY WILL BE EQUAL.

11. (4 points) Sketch a curve that has at least one point at which the curvature is 1 and at least one point at which the curvature is 0. Identify and label such points.



12. (3 points) An object is moving along a straight line. If $\hat{T}(t)$ is the unit tangent vector for the path of the object, what can be said about $\hat{T}'(t)$?

IF $\vec{r}(t)$ DESCRIBES A LINE,
THEN $\hat{\vec{T}}(t)$ IS A CONSTANT VECTOR.

$$\text{Therefore, } \hat{\vec{T}}'(t) = \vec{0}.$$

13. (6 points) At what point do the curves $\vec{r}(t) = t\hat{i} + (1-t)\hat{j} + (3+t^2)\hat{k}$ and $\vec{R}(s) = (3-s)\hat{i} + (s-2)\hat{j} + s^2\hat{k}$ intersect? Find the angle of intersection.

POINT OF INTERSECTION:

$$\begin{aligned} t &= 3-s \\ 1-t &= s-2 \\ 3+t^2 &= s^2 \end{aligned}$$

$$12 = 6s$$

$$\begin{array}{c} s = 2 \\ \downarrow \\ t = 1 \end{array}$$

Angle:

$$\vec{r}'(t) = \hat{i} - \hat{j} + 2t\hat{k}$$

$$\vec{r}'(1) = \hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{R}'(s) = -\hat{i} + \hat{j} + 2s\hat{k}$$

$$\vec{R}'(2) = -\hat{i} + \hat{j} + 4\hat{k}$$

$$\cos \theta = \frac{-1 - 1 + 8}{\sqrt{6} \sqrt{18}}$$

$$\theta \approx 54.7^\circ$$

$$\approx 0.955 \text{ RADIANS}$$

Show all work. Supply explanations when necessary.

1. (8 points) The top of a ladder of length L is sliding down a vertical wall as the base of the ladder is being pulled away from the wall at the rate v_0 . The top of the ladder will come away from the wall at the critical height y_c , where

$$y_c = \sqrt[3]{\frac{2L^2v_0^2}{3g}}$$

Suppose $L = 1.546 \pm 0.001$, $v_0 = 0.84 \pm 0.02$, and $g = 9.72 \pm 0.03$, where standard metric units are being used. Use differentials to approximate the propagated error Δy_c .

$$y_c = \left(\frac{2L^2v_0^2}{3g} \right)^{1/3}$$

$$dy_c = \frac{1}{3} \left(\frac{2L^2v_0^2}{3g} \right)^{-2/3} \left[\frac{4Lv_0^2}{3g} dL + \frac{4L^2v_0}{3g} dv_0 - \frac{2L^2v_0^2}{3g^2} dg \right]$$

$$\Delta y_c \approx \frac{1}{3} \left(\frac{2L^2v_0^2}{3g} \right)^{-2/3} \left[\frac{4Lv_0^2}{3g} \Delta L + \frac{4L^2v_0}{3g} \Delta v_0 - \frac{2L^2v_0^2}{3g^2} \Delta g \right]$$

Plug in $L = 1.546$, $\Delta L = 0.001$, $v_0 = 0.84$, $\Delta v_0 = 0.02$,

$$g = 9.72, \Delta g = 0.03$$

$$\boxed{\Delta y_c \approx 0.007443}$$

$$\boxed{y_c = 0.4872 \pm 0.0074}$$

2. (6 points) Use the definition of differentiable to show that $f(x, y) = x^2 + xy + 2y^2$ is differentiable everywhere.

$$f_x(x, y) = 2x + y$$

$$f_y(x, y) = x + 4y$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= [(x + \Delta x)^2 + (x + \Delta x)(y + \Delta y) + 2(y + \Delta y)^2] - [x^2 + xy + 2y^2]$$

$$= \cancel{x^2} + \underline{2x\Delta x} + \underline{\Delta x^2} + \cancel{xy} + \underline{y\Delta x} + \underline{x\Delta y} + \underline{\Delta x\Delta y} + \cancel{2y^2} + \underline{4y\Delta y} + \cancel{2\Delta y^2}$$

$$- x^2 - xy - 2y^2$$

$$= (2x + y)\Delta x + (x + 4y)\Delta y + (\Delta x + \Delta y)\Delta x + (2\Delta y)\Delta y$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ f_x & f_y & \epsilon_1 & \epsilon_2 \end{matrix}$$

SINCE Δz HAS THE RIGHT FORM

AND $(\epsilon_1, \epsilon_2) \rightarrow (0,0)$ AS $(\Delta x, \Delta y) \rightarrow (0,0)$,

f IS DIFFERENTIABLE.

3. (6 points) Reparameterize the curve with respect to the arc length parameter.

$$\vec{r}(t) = 3 \sin t \hat{i} + 4t \hat{j} + 3 \cos t \hat{k}$$

$$\vec{v}(t) = \vec{r}'(t) = 3 \cos t \hat{i} + 4 \hat{j} - 3 \sin t \hat{k}$$

$$\|\vec{v}(t)\| = \sqrt{9 \cos^2 t + 16 + 9 \sin^2 t} = \sqrt{25} = 5$$

Using
 $t=0$
as base point

$$s(t) = \int_0^t 5 \, d\tau = 5\tau \Big|_0^t = 5t$$

$$s = 5t \Rightarrow t = \frac{s}{5}$$

$$\boxed{\vec{r}(s) = 3 \sin \frac{s}{5} \hat{i} + \frac{4}{5}s \hat{j} + 3 \cos \frac{s}{5} \hat{k}}$$