Math 173 - Quiz 10

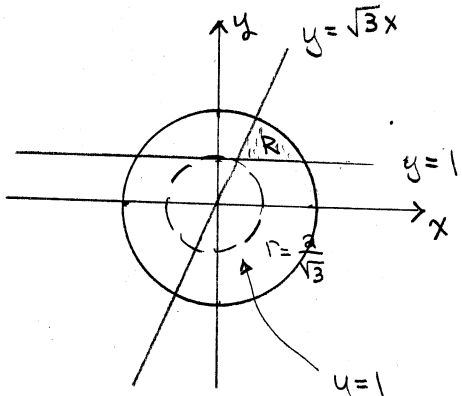
April 21, 2016

Name key

Score

Show all work to receive full credit. Supply explanations when necessary.

1. (5 points) Using polar integration, find the area of the region in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}\,x$.



$y=1$
$y=\sqrt{3}x$
$\Downarrow$
$x=\frac{1}{\sqrt{3}}$

$\sqrt{\frac{1}{3}+1} = \frac{2}{\sqrt{3}}$

$y=1$
$r\sin\theta = 1$
$r = \csc\theta$
$\theta = \csc^{-1} r$

$y=\sqrt{3}x$
$\Downarrow$
$\frac{y}{x} = \sqrt{3}$
$\tan\theta = \sqrt{3}$
$\theta = \frac{\pi}{3}$

$$\int_{r=\frac{2}{\sqrt{3}}}^{r=2} \int_{\theta=\csc^{-1} r}^{\theta=\frac{\pi}{3}} r \, d\theta \, dr$$

$$= \int_{\frac{2}{\sqrt{3}}}^{2} \left(\frac{\pi}{3} r - r\csc^{-1} r\right) dr$$

$$= \frac{\pi}{6} r^2 \Big|_{\frac{2}{\sqrt{3}}}^{2} - \int_{\frac{2}{\sqrt{3}}}^{2} r\csc^{-1} r \, dr$$

$u = \csc^{-1} r \qquad dv = r\,dr$

$du = \frac{-dr}{r\sqrt{r^2-1}} \qquad v = \frac{1}{2} r^2$

$$= \frac{\pi}{6}\left(4 - \frac{4}{3}\right) - \left[\frac{1}{2} r^2 \csc^{-1} r \Big|_{2/\sqrt{3}}^{2} - \int_{\frac{2}{\sqrt{3}}}^{2} \frac{1}{2} \frac{r}{\sqrt{r^2-1}} \, dr\right.$$

$w = r^2 - 1$
$dw = 2r\,dr$

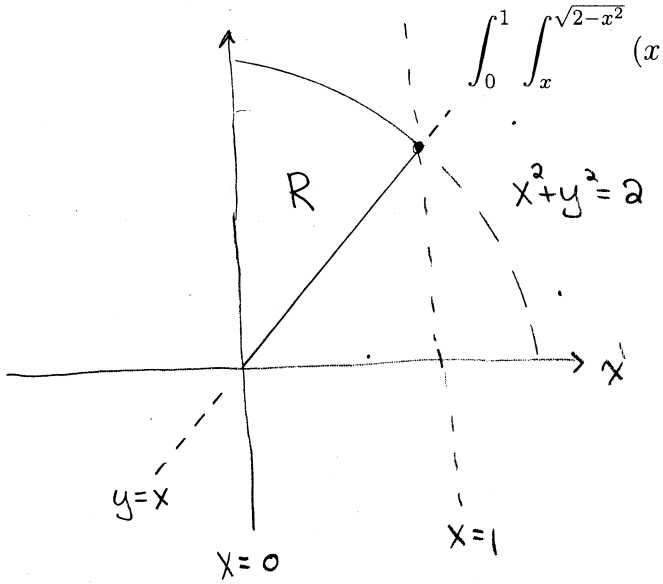
$$= \frac{4\pi}{9} - 2\csc^{-1} 2 + \frac{2}{3}\csc^{-1}\frac{2}{\sqrt{3}} - \int_{1/3}^{3} \frac{1}{4} w^{-1/2} \, dw$$

$$= \frac{4\pi}{9} - 2\csc^{-1} 2 + \frac{2}{3}\csc^{-1}\frac{2}{\sqrt{3}} - \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{\frac{1}{3}}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}} \approx 0.4698$$

2. (5 points) Evaluate by converting to polar coordinates.

$$\int_0^1 \int_x^{\sqrt{2-x^2}} (x+2y)\,dy\,dx$$



$$= \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\pi/2} \frac{1}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta \Big|_0^{\sqrt{2}} d\theta$$

$$= \frac{\sqrt{8}}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos\theta + 2\sin\theta)\, d\theta$$

$$= \frac{\sqrt{8}}{3} \left(\sin\theta - 2\cos\theta\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{\sqrt{8}}{3} \left[\left(1 - 2(0)\right) - \left(\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{\sqrt{8}}{3} \left[1 + \frac{\sqrt{2}}{2}\right]$$

$$= \frac{\sqrt{8}}{3} + \frac{2}{3} \approx 1.609$$