

**Math 173 - Test 1**  
February 18, 2016

Name key Score \_\_\_\_\_

Show all work to receive full credit. Supply explanations where necessary.

1. (5 points) Let  $\vec{v} = \hat{i} + 2\hat{j} - 7\hat{k}$  and  $\vec{w} = 3\hat{i} - \hat{j} - \hat{k}$ . Find a unit vector in the direction of  $3\vec{w} - \vec{v}$ .

$$\begin{aligned} 3\vec{w} - \vec{v} &= 3(3\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} - 7\hat{k}) \\ &= 8\hat{i} - 5\hat{j} + 4\hat{k} \end{aligned}$$
$$\|3\vec{w} - \vec{v}\| = \sqrt{64 + 25 + 16} = \sqrt{105}$$
$$\left\{ \frac{8}{\sqrt{105}}\hat{i} - \frac{5}{\sqrt{105}}\hat{j} + \frac{4}{\sqrt{105}}\hat{k} \right\}$$

2. (2 points) What does it mean for two vectors to be orthogonal?

TWO VECTORS ARE ORTHOGONAL IF THEIR  
DOT PRODUCT IS ZERO.

3. (5 points) Find a 2D unit vector that is normal (perpendicular) to the graph of  $y = x^3$  at the point where  $x = -2$ .

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \text{Slope at } x = -2 \text{ is } m = 12$$
$$\Rightarrow m_{\perp} = -\frac{1}{12}$$

Normal vector is  $12\hat{i} - \hat{j}$

UNIT VECTOR IS

$$\frac{12\hat{i} - \hat{j}}{\sqrt{145}}$$

↑ Mag is  $\sqrt{144+1}$

4. (4 points) Consider the vectors  $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$  and  $\vec{b} = -6\hat{i} - 9\hat{j} + \hat{k}$ . Carefully explain why these vectors are not parallel. Then make changes to  $\vec{b}$  so that  $\vec{a}$  and your new  $\vec{b}$  are parallel.

$$-3\vec{a} = -6\hat{i} + 9\hat{j} - 3\hat{k}$$

$$\vec{b} = -6\hat{i} - 9\hat{j} + \hat{k}$$

MAKE  $\vec{b} = -3\vec{a}$   
 $= -6\hat{i} + 9\hat{j} - 3\hat{k}$

$\vec{b}$  is NOT A

SCALAR MULTIPLE OF  $\vec{a}$ . THEREFORE  $\vec{a}$  AND  $\vec{b}$  ARE NOT PARALLEL.

5. (6 points) Consider the line segment connecting the point  $P(2, 1, 5)$  to the point  $Q(8, -2, 3)$ .

(a) Find the midpoint of the segment.

$$\frac{P+Q}{2} : \left( \frac{2+8}{2}, \frac{1+(-2)}{2}, \frac{5+3}{2} \right) = \left( 5, -\frac{1}{2}, 4 \right)$$

(b) Using the midpoint as your initial point, find a set of parametric equations for the line segment.

$$\vec{PQ} = 6\hat{i} - 3\hat{j} - 2\hat{k}$$

$$\left( 5, -\frac{1}{2}, 4 \right)$$

$$x = 5 + 6t$$

$$y = -\frac{1}{2} - 3t$$

$$z = 4 - 2t$$

$$-\frac{1}{2} \leq t \leq \frac{1}{2}$$

From P to Q

6. (5 points) Find an equation of the plane passing through the point  $(1, 2, 3)$  and normal to the line with symmetric equations

$$\frac{x+9}{2} = \frac{y-3}{-5} = z. \quad \vec{n} = 2\hat{i} - 5\hat{j} + \hat{k}$$

$$2x - 5y + z = 2(1) - 5(2) + 3 = -5$$

$$2x - 5y + z = -5$$

7. (5 points) Let  $\vec{u} = 3\hat{i} + 2\hat{j} - 2\hat{k}$  and  $\vec{v} = 2\hat{i} - 2\hat{j} + 4\hat{k}$ . Find the length of the projection of  $\vec{u}$  onto  $\vec{v}$ .

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 3(2) + 2(-2) + (-2)(4) \\ &= -6 \end{aligned}$$

$$\|\vec{v}\| = \sqrt{4+4+16} = \sqrt{24}$$

↑ ABS VALUE IS  
LENGTH OF PROJECTION

$$\text{LENGTH IS } \frac{6}{\sqrt{24}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

$$\approx 1.225$$

8. (5 points) Find  $\vec{r}(t)$  that satisfies these conditions:

$$\vec{r}'(t) = \frac{1}{1+t^2}\hat{i} + \frac{1}{t^2}\hat{j} + \frac{1}{t}\hat{k}, \quad \vec{r}(1) = 2\hat{i}$$

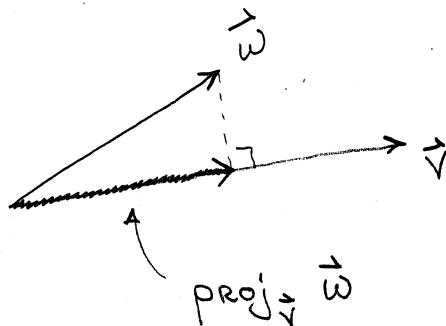
$$\vec{r}(t) = (\tan^{-1} t + c_1)\hat{i} + \left(-\frac{1}{t} + c_2\right)\hat{j} + (\ln|t| + c_3)\hat{k}$$

$$\vec{r}(1) = 2\hat{i} = \left(\frac{\pi}{4} + c_1\right)\hat{i} + (-1 + c_2)\hat{j} + c_3\hat{k}$$

$$\Rightarrow c_1 = 2 - \frac{\pi}{4}, \quad c_2 = 1, \quad c_3 = 0$$

$$\vec{r}(t) = \left(\tan^{-1} t + 2 - \frac{\pi}{4}\right)\hat{i} + \left(1 - \frac{1}{t}\right)\hat{j} + \ln|t|\hat{k}$$

9. (4 points) Sketch a diagram that shows two vectors,  $\vec{v}$  and  $\vec{w}$ , and then show the vector  $\text{proj}_{\vec{v}} \vec{w}$ .



10. (8 points) Consider the vectors  $\vec{u} = -2\hat{i} + \hat{j} - 6\hat{k}$  and  $\vec{v} = -3\hat{i} - 3\hat{j} + 8\hat{k}$ .

(a) Find a vector that is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -6 \\ -3 & -3 & 8 \end{vmatrix} = \hat{i}(8-18) - \hat{j}(-16-18) + \hat{k}(6+3)$$

$$= \boxed{-10\hat{i} + 34\hat{j} + 9\hat{k}}$$

(b) Find the area of the triangle determined by the vectors  $\vec{u}$  and  $\vec{v}$ .

$$A = \frac{1}{2} \|\vec{u} \times \vec{v}\|$$

$$= \frac{1}{2} \sqrt{100 + 1156 + 81} = \boxed{\frac{1}{2} \sqrt{1337} \approx 18.28}$$

11. (8 points) Find the measure of the angle between the planes described by the equations below.

$$3x - 2y + 10z = 0$$

$$5x + 7y + z = 10$$

$$\vec{n}_1 = 3\hat{i} - 2\hat{j} + 10\hat{k}$$

$$\vec{n}_2 = 5\hat{i} + 7\hat{j} + \hat{k}$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} = \frac{15 - 14 + 10}{\sqrt{9+4+100} \sqrt{25+49+1}}$$

$$= \frac{11}{\sqrt{(113)(75)}} \Rightarrow \theta = \cos^{-1} \left( \frac{11}{\sqrt{(113)(75)}} \right)$$

$$\approx \boxed{1.45 \approx 83.14^\circ}$$

12. (12 points) In a field goal attempt on a flat field, a football is kicked off the ground at an angle of  $30^\circ$  to the horizontal with an initial speed of 65 ft/sec.

(a) What horizontal distance does the football travel while it is in the air?

$$\begin{aligned}\vec{r}(t) &= 65 \cos 30^\circ t \hat{i} + (-16t^2 + 65 \sin 30^\circ t) \hat{j} \\ &= \frac{65\sqrt{3}}{2} t \hat{i} + \left(-16t^2 + \frac{65}{2} t\right) \hat{j}\end{aligned}$$

$$\begin{aligned}\frac{65\sqrt{3} (2.03125)}{2} &\approx 114.34 \text{ FT} \\ -16t^2 + \frac{65}{2} t &= 0 \\ t(-16t + \frac{65}{2}) &= 0 \\ t &= \frac{65/2}{16} = 2.03125\end{aligned}$$

- (b) To score a field goal, the ball must clear the cross bar of the goal post, which is 10 ft above the ground. What is the furthest from the goal post the kick can originate and score a field goal?

$$-16t^2 + \frac{65}{2} t = 10 \Rightarrow \text{QUADRATIC FORMULA GIVES}$$

$$t \approx 0.378056 \text{ or}$$

$$t \approx 1.65319$$

$$\frac{65\sqrt{3} (1.65319)}{2} \approx 93.06 \text{ FT}$$

13. (4 points) Find a vector-valued function whose graph is the parabola given by  $y = x^2 + 1$ .

$$\begin{aligned}x &= t \\ y &= t^2 + 1\end{aligned}$$

$\Rightarrow$

$$\vec{r}(t) = t \hat{i} + (t^2 + 1) \hat{j}$$

14. (12 points) An object is moving in such a way that its position at time  $t$  is given by

$$\vec{r}(t) = \sin 3t \hat{i} - \cos 3t \hat{j} + 2t^2 \hat{k}.$$

- (a) Determine the function that gives the speed of the object at time  $t$ .

$$\begin{aligned}\vec{r}'(t) &= 3 \cos 3t \hat{i} + 3 \sin 3t \hat{j} + 4t \hat{k} \\ \text{Speed} &= \|\vec{r}'(t)\| = \sqrt{9 \cos^2 3t + 9 \sin^2 3t + 16t^2} \\ &= \sqrt{9 + 16t^2}\end{aligned}$$

- (b) Find the speed at time  $t = 5$ .

$$\|\vec{r}'(5)\| = \sqrt{9 + 16(25)} = \sqrt{409} \approx 20.22$$

- (c) Find  $\hat{T}(t)$ , the unit tangent vector for  $\vec{r}(t)$ .

$$\hat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{9 + 16t^2}} (3 \cos 3t \hat{i} + 3 \sin 3t \hat{j} + 4t \hat{k})$$

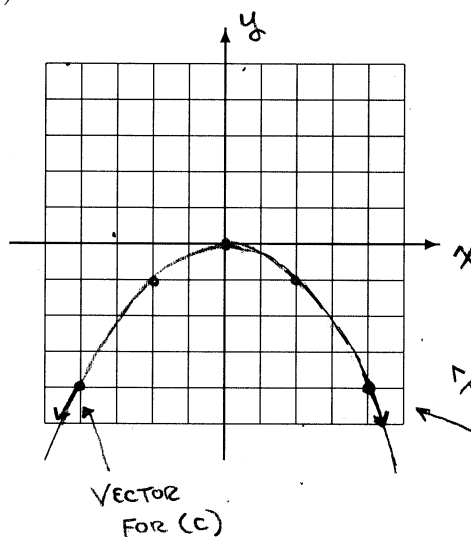
- (d) What would you find if you computed  $\hat{T}(t) \cdot \hat{T}'(t)$ ? Explain.

$$\hat{T}(t) \cdot \hat{T}'(t) = 0$$

Any VECTOR-VALUED FUNCTION OF  
CONSTANT MAGNITUDE IS ORTHOGONAL  
TO ITS DERIVATIVE.

15. (8 points) Consider the vector-valued function  $\vec{r}(t) = 2t\hat{i} - t^2\hat{j}$ .

(a) Sketch the graph of  $\vec{r}(t)$ .



$$x = 2t \rightarrow t = \frac{x}{2}$$

$$y = -t^2 = -\frac{x^2}{4}$$

(b) Without computing  $\hat{T}(t)$ , sketch the vector  $\hat{T}(2)$  on your graph.

WITH INCREASING  $t$ , POINTS ALONG GRAPH MOVE RIGHT.

(c) How would your sketch of  $\hat{T}(2)$  be different if the first component of  $\vec{r}(t)$  was  $-2t$  rather than  $2t$ ? Explain.

IN THIS CASE, MOTION ALONG THE CURVE

WOULD BE IN THE OPPOSITE DIRECTION.

$\hat{T}(2)$  WOULD HAVE ITS TAIL AT  $(-4, -4)$  AND

16. (7 points) Let  $\vec{r}(t) = -7\cos t\hat{i} - 7\sin t\hat{j} + t\hat{k}$ . Compute  $\hat{N}(t)$ .

POINT DOWN & LEFT.

$$\vec{r}'(t) = 7\sin t\hat{i} - 7\cos t\hat{j} + \hat{k}$$

$$\|\vec{r}'(t)\| = \sqrt{49\sin^2 t + 49\cos^2 t + 1} = \sqrt{50}$$

$$\hat{T}(t) = \frac{1}{\sqrt{50}} (7\sin t\hat{i} - 7\cos t\hat{j} + \hat{k})$$

$$\hat{T}'(t) = \frac{1}{\sqrt{50}} (7\cos t\hat{i} + 7\sin t\hat{j})$$

$$\|\hat{T}'(t)\| = \frac{1}{\sqrt{50}} \sqrt{49} = \frac{7}{\sqrt{50}}$$

$$\hat{N}(t) = \cos t\hat{i} + \sin t\hat{j}$$