

Math 173 - Test 2  
March 31, 2016

Name key Score \_\_\_\_\_

Show all work. Supply explanations when necessary.

YOU MUST WORK INDIVIDUALLY ON THIS TEST.

1. (10 points) Determine the limit or show that it does not exist.

$$(a) \lim_{(x,y) \rightarrow (2,0)} \frac{(x-y)^2 + 2(x-y) - x^2 - 2x}{y}$$

$$= \lim_{(x,y) \rightarrow (2,0)} \frac{x^2 - 2xy + y^2 + 2x - 2y - x^2 - 2x}{y}$$

$$= \lim_{(x,y) \rightarrow (2,0)} \frac{-2xy + y^2 - 2y}{y} = \lim_{(x,y) \rightarrow (2,0)} (-2x + y - 2)$$

$$= \boxed{-6}$$

$$(b) \lim_{(x,y) \rightarrow (1,2)} \frac{xy^3 - 8x^2}{xy^2 - 4}$$

$$x=1: \lim_{y \rightarrow 2} \frac{y^3 - 8}{y^2 - 4} = \lim_{y \rightarrow 2} \frac{y^3 + 2y + 4}{y+2} = \frac{12}{4} = 3$$

$$y=2: \lim_{x \rightarrow 1} \frac{8x - 8x^2}{4x - 4} = \lim_{x \rightarrow 1} \frac{8x(1-x)}{4(x-1)} = \lim_{x \rightarrow 1} -\frac{8x}{4} = -2$$

} LIMIT  
DNE

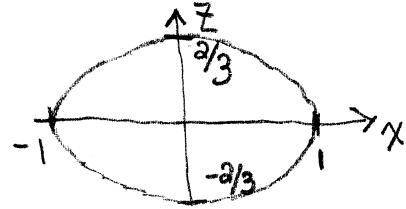
2. (5 points) Consider the surface described by the equation  $4x^2 - y^2 + 9z^2 = 4$ .

- (a) Fix a value for one of the variables and draw a good sketch of the corresponding level curve.

$$y=0 : 4x^2 + 9z^2 = 4$$

$$x^2 + \frac{z^2}{4/9} = 1$$

Ellipse with vertices  $(\pm 1, 0)$  &  $(0, \pm \frac{2}{3})$



- (b) Fix a value for one of the other variables and briefly describe the corresponding level curve.

$$z=0 : 4x^2 - y^2 = 4$$

$$x^2 - \frac{y^2}{4} = 1$$

Hyperbola centered at  $(0,0)$

Vertices at  $(\pm 1, 0)$ ,

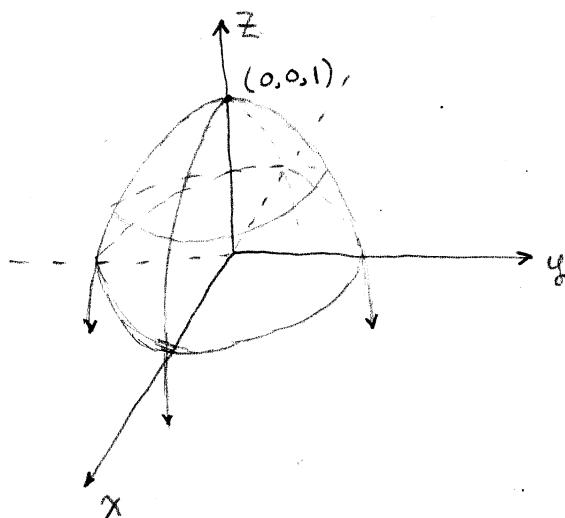
opening left/right.

- (c) Identify the surface.

$$4x^2 + 9z^2 = 4 + y^2$$

Hyperboloid of ONE SHEET

3. (5 points) Draw a reasonably good sketch of the graph of  $f(x, y) = 1 - x^2 - y^2$  and identify the surface.



Paraboloid,

opening downward,

circular level

curves centered on

$z$ -axis,

vertex at  $(0,0,1)$

4. (8 points) Use differentials to estimate the change in  $f(x, y, z) = x^2 \ln(5yz + 1)$  as  $(x, y, z)$  changes from  $(2, 1, 3)$  to  $(1.99, 1.02, 3.05)$ .

$$d\omega = f_x dx + f_y dy + f_z dz = 2x \ln(5yz+1) dx + \frac{5x^2z}{5yz+1} dy + \frac{5x^2y}{5yz+1} dz$$

$$\Delta\omega \approx 2x \ln(5yz+1) \Delta x + \frac{5x^2z}{5yz+1} \Delta y + \frac{5x^2y}{5yz+1} \Delta z$$

$$x=2, y=1, z=3$$

$$\Delta x = -0.01, \Delta y = 0.02, \Delta z = 0.05$$

$$\Delta\omega \approx (4 \ln 16)(-0.01) + \frac{60}{16}(0.02) + \frac{20}{16}(0.05)$$

$$\approx 0.0266$$

5. (8 points) Suppose the equation  $xyz + e^{-xz^2} + 5 \cos 2xy = 6$  implicitly defines  $z$  as a function of  $x$  and  $y$ . Find  $\partial z / \partial y$  at the point  $(0, 3, 1)$ .

$$F(x, y, z) = xyz + e^{-xz^2} + 5 \cos 2xy - 6$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(xz - 10x \sin 2xy)}{xy - 2xz e^{-xz^2}}$$

TECHNICALLY NOT DEFINED WHEN  $X=0$ ,

BUT THE DISCONT IS REMOVEABLE.

$$\frac{\partial z}{\partial y} = \frac{-z + 10 \sin 2xy}{y - 2ze^{-xz^2}}$$

$$\text{WHEN } (x, y, z) \rightarrow (0, 3, 1) \rightarrow \frac{\partial z}{\partial y} = \frac{-1 + 0}{3 - 0} = -1$$

6. (10 points) Consider the function  $g(x, y, z) = x \tan^{-1}(y/z)$ .

- (a) Find the directional derivative of  $g$  at the point  $(1, 2, -2)$  in the direction of  $\vec{w} = \hat{i} + \hat{j} - \hat{k}$ .

$$\nabla g(x, y, z) = \tan^{-1}\left(\frac{y}{z}\right) \hat{i} + \frac{xz^{-1}}{1 + (\frac{y}{z})^2} \hat{j} + \frac{-xyz^{-2}}{1 + (\frac{y}{z})^2} \hat{k}$$

$$\nabla g(1, 2, -2) = \tan^{-1}(-1) \hat{i} - \frac{1}{4} \hat{j} - \frac{1}{4} \hat{k} = -\frac{\pi}{4} \hat{i} - \frac{1}{4} \hat{j} - \frac{1}{4} \hat{k}$$

$$\vec{w} = \hat{i} + \hat{j} - \hat{k} \Rightarrow \|\vec{w}\| = \sqrt{3}$$

$$D_{\vec{w}} g(1, 2, -2) = \frac{1}{\sqrt{3}} \left( -\frac{\pi}{4} - \frac{1}{4} + \frac{1}{4} \right) = -\frac{\pi}{4\sqrt{3}} \approx -0.453$$

- (b) At the point  $(1, 2, -2)$ , what is the direction of steepest descent (maximum decrease) of  $g$ ?

$$\text{Opposite } \nabla g(1, 2, -2)$$

$$= \underline{\frac{\pi}{4} \hat{i} + \frac{1}{4} \hat{j} + \frac{1}{4} \hat{k}}$$

7. (6 points) Let  $g(x, y, z) = x \cos(y^2 z + yz^2) - x^2 e^{y^2 z} + x\sqrt{4y + 2z}$ . Suppose you'd like to determine the mixed partial derivative  $g_{yzx}$ . Which mixed partial derivative is equal to  $g_{yzx}$ , but would be much simpler to compute? Compute that partial derivative.

$g_{xxyz}$  SHOULD BE SIMPLEST

BECAUSE 1<sup>ST</sup> AND LAST TERMS BECOME ZERO

AFTER  $g_{xx}$ . ( $g_{xxyz}$  LOOKS EQUALLY SIMPLE.)

$$g_{xx} = -2e^{y^2 z}$$

$$g_{xxy} = -4yz e^{y^2 z}$$

$$g_{xxyz} = -4ye^{y^2 z}$$

$$- 4y^3 ze^{y^2 z}$$

8. (10 points) Find and classify the critical points of  $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$ .

$$f_x(x, y) = 6x^2 + y^2 + 10x = 0$$

$$f_y(x, y) = 2xy + 2y = 0 \Rightarrow 2y(x+1) = 0$$

$$y=0 \quad x=-1$$

$$\downarrow \qquad \downarrow$$

$$6x^2 + 10x = 0 \quad y^2 = 4$$

$$2x(3x+5) = 0 \quad y = \pm 2$$

$$x=0, \quad x = -\frac{5}{3}$$

$$(0,0), \left(-\frac{5}{3}, 0\right) \quad (-1, 2), (-1, -2)$$

$$D(x, y) = \begin{vmatrix} 12x + 10 & 2y \\ 2y & 2x + 2 \end{vmatrix} = (12x + 10)(2x + 2) - 4y^2$$

$$D(0,0) = 0 \text{ AND } f_{xx}(0,0) = 10 > 0 \Rightarrow f(0,0) = 0 \text{ IS A REL MIN}$$

$$D\left(-\frac{5}{3}, 0\right) = \frac{40}{3} \text{ AND } f_{xx}\left(-\frac{5}{3}, 0\right) = -10 < 0 \Rightarrow f\left(-\frac{5}{3}, 0\right) = \frac{125}{27} \text{ IS A REL MAX.}$$

$$D(-1, 2) = -16 \Rightarrow (-1, 2, f(-1, 2)) = (-1, 2, 3) \text{ IS A SADDLE PT.}$$

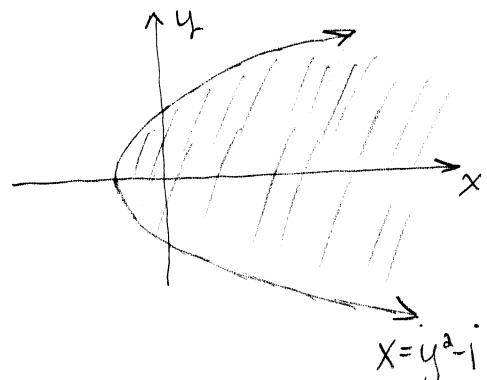
$$D(-1, -2) = -16 \Rightarrow (-1, -2, f(-1, -2)) = (-1, -2, 3) \text{ IS A SADDLE PT.}$$

9. (8 points) Consider the function  $h(x, y) = \sqrt{1 + x - y^2}$ .

(a) What is the domain of  $h$ ?

$$\begin{aligned} 1 + x - y^2 &\geq 0 \\ \Rightarrow x &\geq y^2 - 1 \end{aligned}$$

$$\{(x, y) : x \geq y^2 - 1\}$$



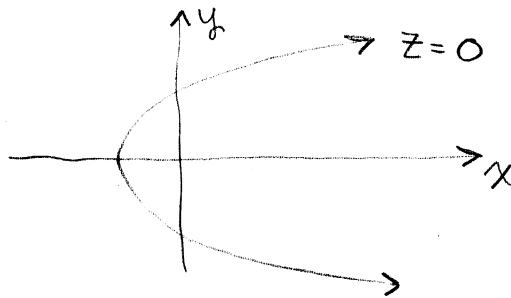
(b) Discuss the continuity of  $h$ .

$h$  is continuous.

ON ITS ENTIRE DOMAIN.

(c) Sketch the level curve  $h(x, y) = 0$ .

$$\begin{aligned} 1 + x - y^2 &= 0 \\ \Rightarrow x &= y^2 - 1 \end{aligned}$$



(d) The graph of  $h$  is one-half of one of the quadric surfaces that we are familiar with.  
Describe the graph of  $h$ .

$$z^2 = 1 + x - y^2$$



$$y^2 + z^2 - 1 = x$$



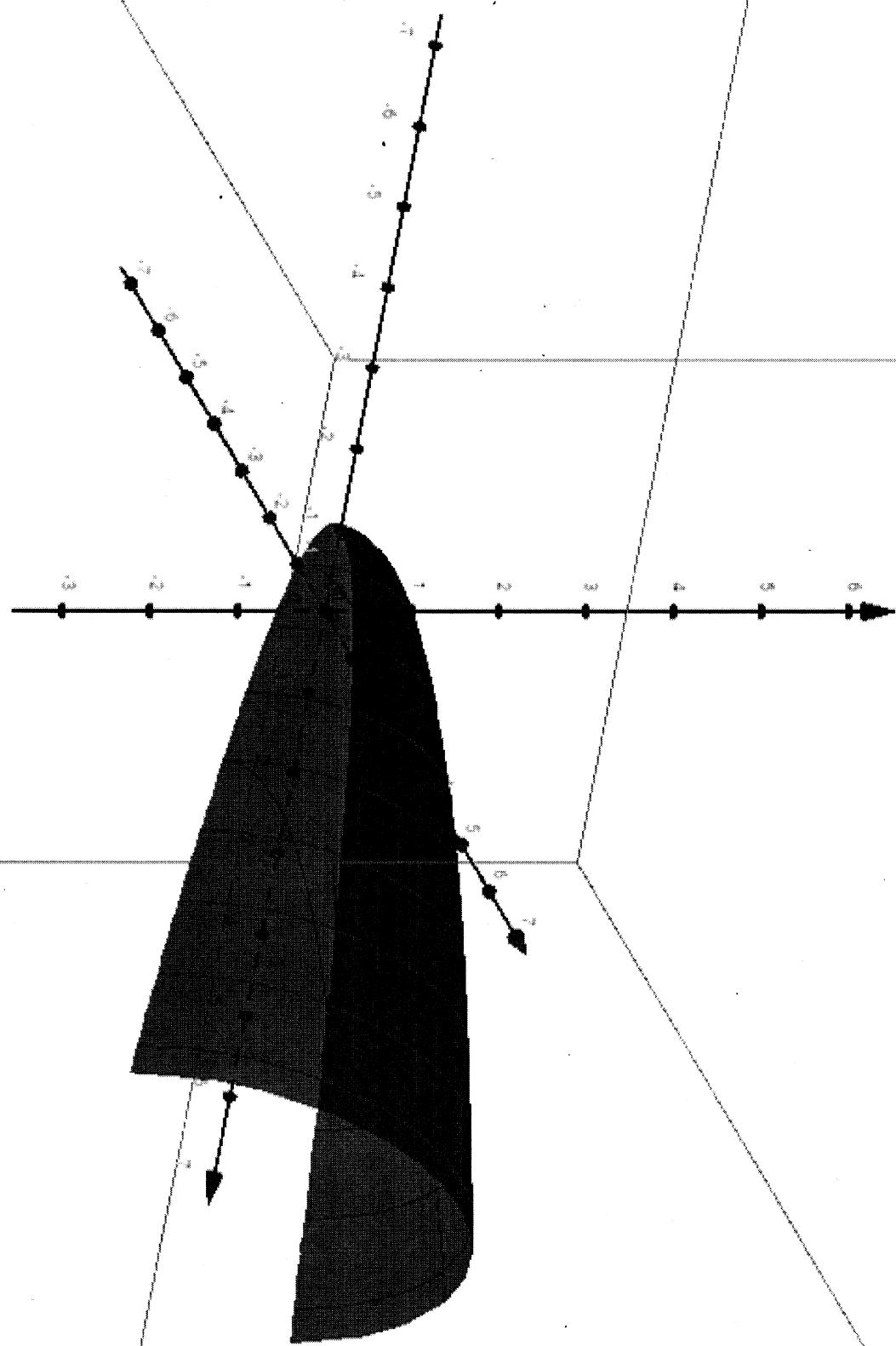
THIS DESCRIBES A

CIRCULAR PARABOLOID OPENING  
UP X-AXIS WITH VERTEX AT (-1, 0, 0).

THE GRAPH OF  $h$  IS THE

UPPER HALF OF THE  
PARABOLOID

(SEE NEXT PAGE.)



10. (10 points) A surface is described by the equation  $z + 1 = xe^y \cos z$ . Find equations for the tangent plane and normal line through  $(1, 0, 0)$ .

$$F(x, y, z) = xe^y \cos z - z$$

Our SURFACE IS THE LEVEL SURFACE  $F(x, y, z) = 1$ .

$$\vec{\nabla} F(x, y, z) = (e^y \cos z) \hat{i} + (xe^y \cos z) \hat{j} - (xe^y \sin z + 1) \hat{k}$$

$$\vec{n} = \vec{\nabla} F(1, 0, 0) = \hat{i} + \hat{j} - \hat{k}$$

TANGENT PLANE:

$$x + y - z = 1$$

NORMAL LINE:

$$x = t + 1$$

$$y = t$$

$$z = -t$$

11. (10 points) At what point does the function  $f(x) = e^x$  have the greatest curvature?  
 (Hint: Compute the curvature function and find its maximum value.)

$$\vec{r}(t) = t\hat{i} + e^t\hat{j}, \quad \vec{v}(t) = \hat{i} + e^t\hat{j}, \quad \vec{a}(t) = e^t\hat{j}$$

$$K(t) = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3} \quad \vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & e^t & 0 \\ 0 & e^t & 0 \end{vmatrix} = e^t \hat{k}$$

$$K(t) = \frac{e^t}{(1+e^{at})^{3/2}}$$

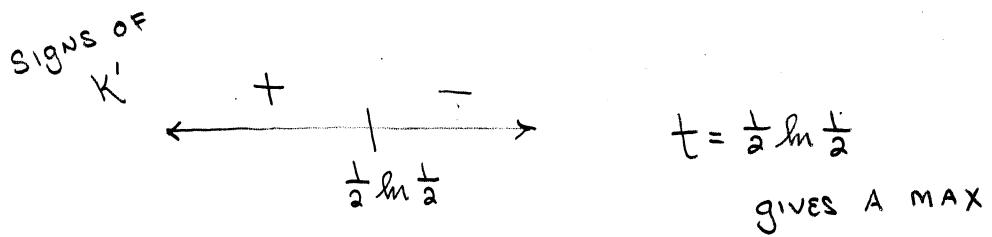
$$K'(t) = \frac{(1+e^{at})^{3/2}(e^t) - e^t \left(\frac{3}{2}\right)(1+e^{at})^{1/2}(ae^{at})}{(1+e^{at})^3}$$

$$= \frac{(1+e^{at})^{1/2}e^t \left[ (1+e^{at}) - 3e^{at} \right]}{(1+e^{at})^3} = \frac{e^t(1-2e^{at})}{(1+e^{at})^{5/2}}$$

$$K'(t) = 0 \Rightarrow 1-2e^{at} = 0$$

$$e^{at} = \frac{1}{2}$$

$$t = \frac{1}{a} \ln \frac{1}{2} \approx -0.3466$$



POINT OF MAX CURVATURE IS

$$\left( \frac{1}{2} \ln \frac{1}{2}, \sqrt{\frac{1}{a}} \right)$$

$$L(x, y) = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

12. (10 points) Suppose we are given the two functions

$$f_x(x, y) = y + 1$$

$$f(x, y) = xy + x - 4y - 10,$$

$$g(x, y) = xy - 6x + y - 8$$

$$g_x(x, y) = y - 6$$

$$g_y(x, y) = x + 1$$

$f_y(x, y) = x - 4$  and we wish to solve the system of nonlinear equations  $f(x, y) = 0$ ,  $g(x, y) = 0$ . We can approximate a solution using linearizations and Newton's method.

- (a) Let  $(x_0, y_0) = (-1, -2)$  be our initial guess at the solution. Let  $f_0(x, y)$  and  $g_0(x, y)$  be the linearizations of  $f$  and  $g$  at the point  $(x_0, y_0)$ . Find  $f_0(x, y)$  and  $g_0(x, y)$ .

$$\begin{aligned} f_0(x, y) &= f(-1, -2) + f_x(-1, -2)(x+1) + f_y(-1, -2)(y+2) \\ &= -1 + (-1)(x+1) + (-5)(y+2) = -x - 5y - 12 \end{aligned}$$

$$\begin{aligned} g_0(x, y) &= g(-1, -2) + g_x(-1, -2)(x+1) + g_y(-1, -2)(y+2) \\ &= -2 + (-8)(x+1) + 0(y+2) = -8x - 10 \end{aligned}$$

- (b) Solve the linear system of equations

$$f_0(x, y) = 0, \quad g_0(x, y) = 0.$$

$$-x - 5y = 12$$

$$-8x = 10$$

$$x = -\frac{5}{4} = -1.25$$

$$y = -\frac{1}{5}(12 - \frac{5}{4}) = \frac{43}{20} = 2.15$$

- (c) Let  $(x_1, y_1)$  be the solution of the linear system in part (b). It represents an improved guess at the solution of the original system. Compute  $f(x_1, y_1)$  and  $g(x_1, y_1)$ .

$$(x_1, y_1) = (-1.25, -2.15)$$

$$f(-1.25, -2.15) = 0.0375 \quad g(-1.25, -2.15) = 0.0375$$

- (d) (Extra Credit 4 pts) On a separate sheet of paper, use  $(x_1, y_1)$  in place of  $(x_0, y_0)$  and repeat the steps above to further improve our guess at the solution.

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[%i62) f:x*y+x-4*y-10$ g:x*y-6*x+y-8$  
[%i64) fx:diff(f,x)$ fy:diff(f,y)$  
[%i66) gx:diff(g,x)$ gy:diff(g,y)$  
[%i68) x0:-1$  
y0:-2$  
f0:at(f, [x=x0, y=y0]) + at(fx, [x=x0, y=y0]) * (x-x0)  
+ at(fy, [x=x0, y=y0]) * (y-y0);  
g0:at(g, [x=x0, y=y0]) + at(gx, [x=x0, y=y0]) * (x-x0)  
+ at(gy, [x=x0, y=y0]) * (y-y0);  
(%o70) -5(y+2)-x-2  
(%o71) -8(x+1)-2  
[%i72) solve([f0=0, g0=0], [x, y]);  
(%o72) [[x=-5/4, y=-43/20]]  
[%i73) at(f, [x=-5/4, y=-43/20]); at(g, [x=-5/4, y=-43/20]);  
(%o73) 3/80  
(%o74) 3/80  
[%i75) x1:-5/4$  
y1:-43/20$  
f1:ratsimp(at(f, [x=x1, y=y1]) + at(fx, [x=x1, y=y1]) * (x-x1)  
+ at(fy, [x=x1, y=y1]) * (y-y1));  
g1:ratsimp(at(g, [x=x1, y=y1]) + at(gx, [x=x1, y=y1]) * (x-x1)  
+ at(gy, [x=x1, y=y1]) * (y-y1));  
(%o77) -(420 y+92 x+1015)/80  
(%o78) -(20 y+652 x+855)/80  
[%i79) solve([f1=0, g1=0], [x, y]);  
(%o79) [[x=-847/680, y=-7289/3400]]  
[%i80) float(%);  
(%o80) [[x=-1.245588235294118, y=-2.143823529411765]]
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[ (%i81) at(f, [x=-847/680, y=-7289/3400]); at(g, [x=-847/680, y=-7289/3400]);  
(%o81)  $\frac{63}{2312000}$   
(%o82)  $\frac{63}{2312000}$ 
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