Math 173 - Test 3a

April 28, 2016

Name key

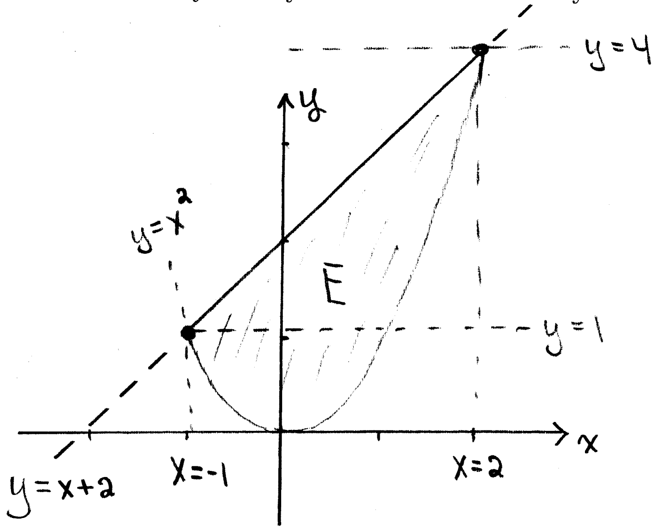
Score ____________

Show all work to receive full credit. Supply explanations where necessary. This portion of the test is due Tuesday, May 3. YOU MUST WORK INDIVIDUALLY ON THIS TEST—YOU WILL NOT BE GIVEN ANY CREDIT FOR GROUP WORK.

1. (12 points) Let E be the plane region between the graphs of $y = x^2$ and $y = x + 2$. Sketch the region E, write the iterated integrals (both orders of integration) associated with the double integral

$$\iint_E (xy + 5)\, dA,$$

and evaluate either one of the iterated integrals. Carry out the integration by hand, but you may use a CAS to check your work.



$$\int_{-1}^{2}\int_{x^2}^{x+2} (xy+5)\,dy\,dx = \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} (xy+5)\,dx\,dy + \int_1^4\int_{y-2}^{\sqrt{y}} (xy+5)\,dx\,dy$$

$x^2 = x+2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2,\ x = -1$

$$\int_{-1}^{2}\int_{x^2}^{x+2} (xy+5)\,dy\,dx = \int_{-1}^{2} \left. \frac{xy^2}{2} + 5y \right|_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} \left[\frac{x^3+4x^2+4x}{2} + 5(x+2) - \frac{x^5}{2} - 5x^2 \right] dx$$

$$= \int_{-1}^{2} \left(-\frac{x^5}{2} + \frac{x^3}{2} - 3x^2 + 7x + 10 \right) dx = \left. -\frac{x^6}{12} + \frac{x^4}{8} - x^3 + \frac{7x^2}{2} + 10x \right|_{-1}^{2}$$

$$= \frac{68}{3} - \left(-\frac{131}{24}\right) = \boxed{\frac{225}{8}}$$

2. (12 points) Use the method of Lagrange multipliers to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$\underbrace{\frac{x^2}{8} + \frac{y^2}{2}}_{g(x,y)} = 1.$$

$f_x(x,y) = y$

$f_y(x,y) = x$

$g_x(x,y) = \frac{x}{4}$

$g_y(x,y) = y$

$$\left.\begin{aligned} y &= \tfrac{1}{4}\lambda x \\ x &= \lambda y \end{aligned}\right\}$$ CAN'T HAVE $x = y = 0$

$\frac{x^2}{8} + \frac{y^2}{2} = 1$

SO $\frac{4y}{x} = \lambda = \frac{x}{y} \Rightarrow 4y^2 = x^2$

$\frac{4y^2}{8} + \frac{y^2}{2} = 1 \Rightarrow y^2 = 1$

$\Rightarrow y = \pm 1$

$y = +1 \Rightarrow x = \pm 2$

$y = -1 \Rightarrow x = \pm 2$

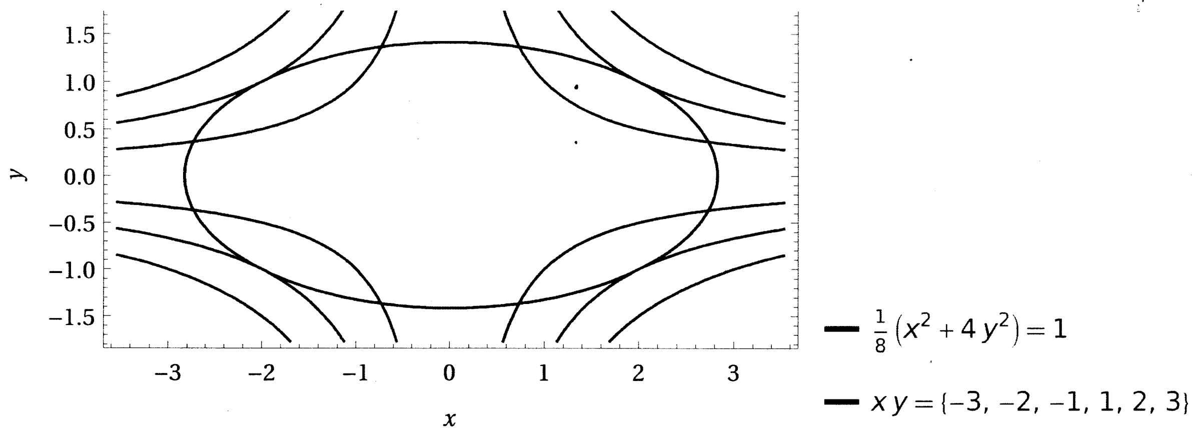
FOUR CRIT PTS: $(1,2), (-1,2), (1,-2), (-1,-2)$

MAX VALUE IS $f(\pm 1, \pm 2) = 2$

MIN VALUE IS $f(\pm 1, \mp 2) = -2$

Follow-up: Sketch the graph of the constraint equation as well as several of the level curves of $f(x, y)$ (i.e., $xy = \pm k$). Mark your critical points on the graph. (If you sketch these graphs by hand, do so carefully! You may use technology such as GeoGebra if you prefer.)

SEE NEXT PAGE.



Computed by Wolfram|Alpha

3. (8 points) Sketch the region of integration. Then write the iterated integral with the reversed order of integration. You need not evaluate the integral.

$$\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y \, dx \, dy$$

$0 \leq y \leq \sqrt{2}$

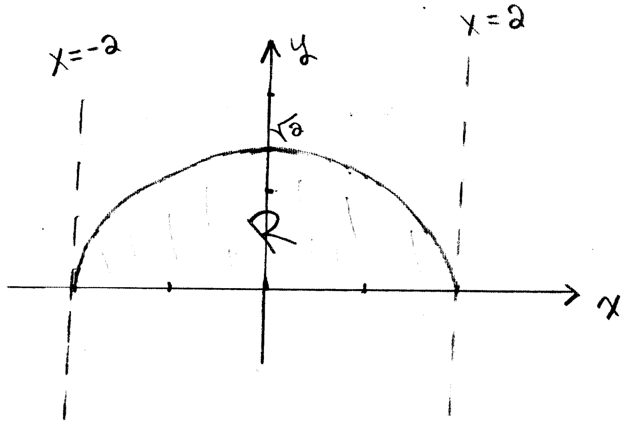
$-\sqrt{4-2y^2} \leq x \leq \sqrt{4-2y^2}$.

$x^2 = 4-2y^2$

$\Downarrow$

$\underbrace{x^2 + 2y^2 = 4}$

Ellipse



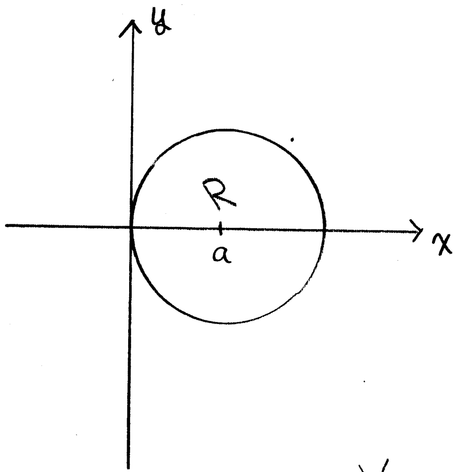
$$\boxed{\int_{-2}^{2} \int_{0}^{\sqrt{\frac{4-x^2}{2}}} y \, dy \, dx}$$

4. (12 points) Find the volume of the space region bounded above by the cone $\overbrace{z^2 = x^2+y^2}^{z = \sqrt{x^2+y^2}}$ and below by the region that lies inside the curve $x^2 + y^2 = 2ax$, where a represents a positive real number. Carry out the integration by hand, but you may use a CAS to check your work.

$$x^2+y^2 = 2ax$$
$$x^2 - 2ax + a^2 + y^2 = a^2$$
$$(x-a)^2 + y^2 = a^2$$

OR

$$r^2 = 2ar\cos\theta$$
$$r = 2a\cos\theta$$



$$\text{Volume} = \iint_R \sqrt{x^2+y^2}\, dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \sqrt{r^2}\, r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2\, dr\, d\theta$$

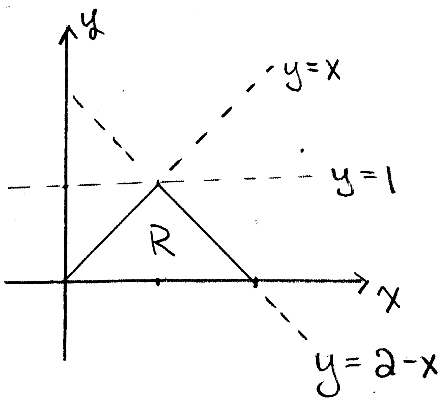
$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8a^3 \cos^3\theta\, d\theta = \frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\, d\theta$$

$$u = \sin\theta$$
$$du = \cos\theta\, d\theta$$

$$\frac{8a^3}{3}\int_{-1}^{1} (1-u^2)\, du = \frac{8a^3}{3}\left(u - \frac{u^3}{3}\right)\Bigg|_{-1}^{1}$$

$$= \frac{8a^3}{3}\left(\frac{2}{3} + \frac{2}{3}\right) = \frac{32a^3}{9}$$

5. (16 points) A thin plate has the shape of a triangle with vertices at $(0,0)$, $(1,1)$ and $(2,0)$. The density of the plate at the point (x,y) is given by $\rho(x,y) = 2x + y + 1$. Find the center of mass of the plate. Compute the mass by hand, but you may use a CAS for the moments.



$$\text{Mass} = \iint_R (2x+y+1)\,dA$$

$$= \int_{y=0}^{y=1} \int_{x=y}^{x=2-y} (2x+y+1)\,dx\,dy$$

$$= \int_0^1 x^2 + xy + x \Big|_y^{2-y} dy$$

$$= \int_0^1 (4-4y+y^2+2y-y^2+2-y-y^2-y^2-y)\,dy$$

$$= \int_0^1 (6-4y-2y^2)\,dy = 6-2-\frac{2}{3} = \frac{10}{3}$$

$$M_y = \iint_R x(2x+y+1)\,dA = \frac{11}{3}$$

$$M_x = \iint_R y(2x+y+1)\,dA = \frac{7}{6}$$

Center of mass is

$$\left(\frac{11/3}{10/3}, \frac{7/6}{10/3}\right) = \left(\frac{11}{10}, \frac{7}{20}\right)$$

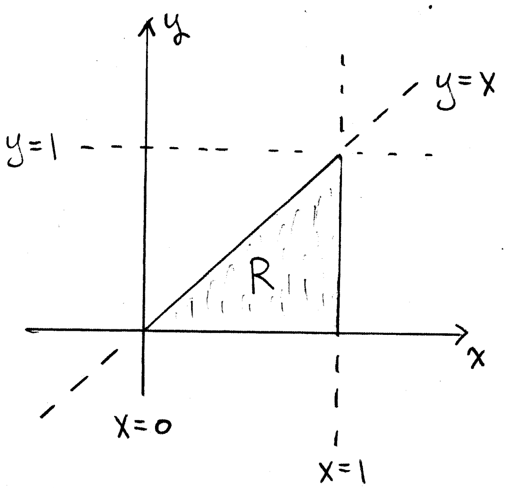
Math 173 - Test 3b
May 3, 2016

Name key
Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Sketch the region of integration and evaluate $\int_0^1 \int_y^1 \frac{e^x}{x}\, dx\, dy$.

   (Hint: Reverse the order of integration.)



$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{e^x}{x}\, dy\, dx$$

$$= \int_0^1 \frac{e^x}{x}\, y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 e^x\, dx$$

$$= e^x \Big|_0^1 = \boxed{e-1}$$

2. (10 points) Use the method of Lagrange multipliers to find the point on the plane $x + 2y + 3z = 6$ that is closest to the origin. (Hint: Minimize $f(x, y, z) = x^2 + y^2 + z^2$, which gives the square of the distance from the origin.)

$f(x,y,z) = x^2 + y^2 + z^2$

$g(x,y,z) = x + 2y + 3z = 6$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda 2 \\ 2z &= \lambda 3 \end{aligned}\right\} \quad \begin{aligned} x &= \lambda/2 \\ y &= \lambda \\ z &= 3\lambda/2 \end{aligned}$$

$x + 2y + 3z = 6$

$$\frac{\lambda}{2} + 2\lambda + 3\left(\frac{3\lambda}{2}\right) = 6$$

$$7\lambda = 6$$

$$\Downarrow$$

$$\lambda = \frac{6}{7}$$

$$\Downarrow$$

SINGLE CRIT. PT. IS

$$\left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)$$

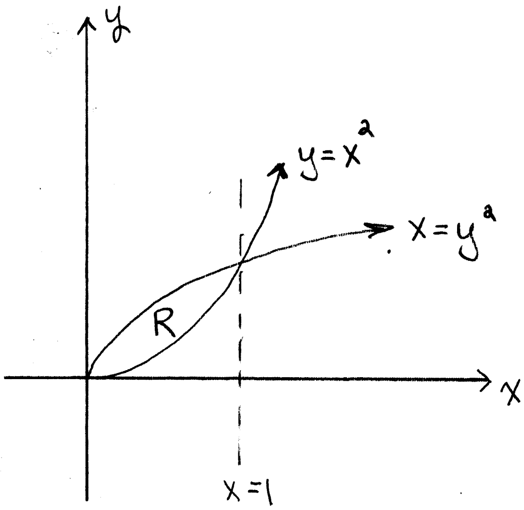
$x = \frac{6/7}{2}$

$y = 6/7$

$z = \frac{3(6/7)}{2}$

THERE IS NO POINT ON THE PLAN THAT IS FARTHEST FROM THE ORIGIN, SO THIS POINT MUST GIVE A MINIMUN DISTANCE.

3. (10 points) Find the average value of $f(x,y) = x^2 + y^2$ over the region bounded by the graphs of $y = x^2$ and $x = y^2$.



AREA OF R

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} dy\, dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\, dx$$

$$= \frac{2}{3} x^{3/2} - \frac{1}{3} x^3 \Big|_0^1$$

$$= \frac{1}{3}$$

AVERAGE VALUE

$$= \frac{1}{1/3} \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} (x^2 + y^2)\, dy\, dx$$

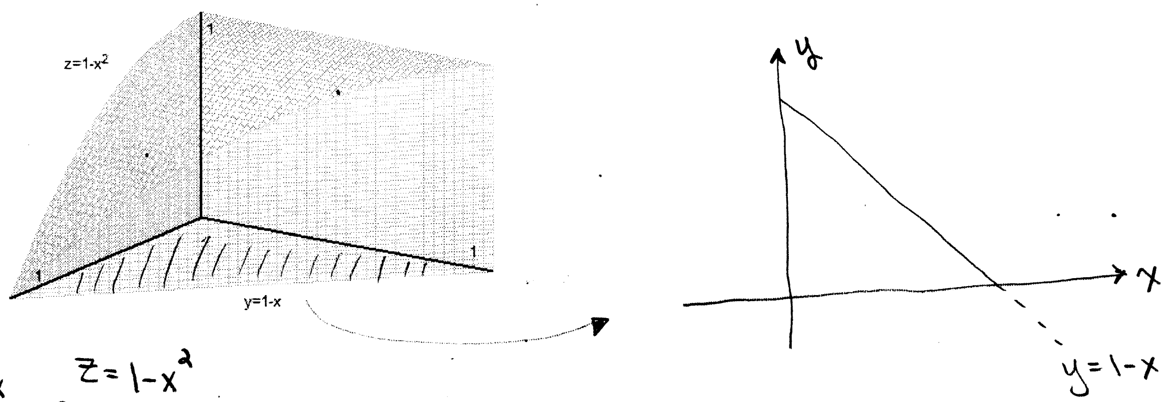
$$= 3 \int_0^1 \left(y x^2 + \frac{1}{3} y^3 \right) \Big|_{x^2}^{\sqrt{x}} dx$$

$$= 3 \int_0^1 \left(x^{5/2} + \frac{1}{3} x^{3/2} - x^4 - \frac{1}{3} x^6 \right) dx$$

$$= 3 \left(\frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} \right) = 3 \left(\frac{6}{35} \right)$$

$$= \boxed{\frac{18}{35}}$$

4. (10 points) Let S be the space region bounded by the surfaces $x = 0$, $y = 0$, $y = 1 - x$, and $z = 1 - x^2$ (see the figure below). Evaluate the triple integral $\displaystyle\iiint_S z\, dV$.



$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x^2} z\, dz\, dy\, dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \frac{1}{2}(1-x^2)^2\, dy\, dx = \int_0^1\int_0^{1-x}\left(\frac{1}{2} - x^2 + \frac{1}{2}x^4\right) dy\, dx$$

$$= \int_0^1 \left(\frac{1}{2} - x^2 + \frac{1}{2}x^4\right)(1-x)\, dx = \int_0^1 \left(\frac{1}{2} - x^2 + \frac{1}{2}x^4 - \frac{1}{2}x + x^3 - \frac{1}{2}x^5\right) dx$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{10} - \frac{1}{4} + \frac{1}{4} - \frac{1}{12} = \boxed{\frac{11}{60}}$$