

Math 173 - Test 4
May 16, 2016

Name key Score _____

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) Let $\vec{u} = 3\hat{i} - 5\hat{k}$ and $\vec{w} = -\hat{i} + 2\hat{j} + 3\hat{k}$.

- (a) Find a vector of magnitude 2 that has the direction of \vec{w} .

$$\|\vec{w}\| = \sqrt{1+4+9} = \sqrt{14}$$

$$\frac{2\vec{w}}{\sqrt{14}} = \left(-\frac{2}{\sqrt{14}}\hat{i} + \frac{4}{\sqrt{14}}\hat{j} + \frac{6}{\sqrt{14}}\hat{k} \right)$$

- (b) Find the measure of the angle between \vec{u} and \vec{w} .

$$\vec{u} \cdot \vec{w} = -3 + 0 - 15 = -18$$

$$\|\vec{u}\| = \sqrt{9+25} = \sqrt{34}$$

$$\|\vec{w}\| = \sqrt{14}$$

$$\cos \theta = \frac{-18}{\sqrt{34}\sqrt{14}}$$

$$\Rightarrow \theta \approx 2.54$$

$$\theta \approx 145.6^\circ$$

- (c) Find the projection of \vec{w} onto \vec{u} .

$$\text{proj}_{\vec{u}} \vec{w} = \frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \frac{-18}{34} \vec{u} = -\frac{9}{17} \vec{u} = \left(-\frac{27}{17}\hat{i} + \frac{45}{17}\hat{k} \right)$$

2. (10 points) Write a direction vector for each line. Then show that the lines are parallel.

$$L_1: x = 2 + 3t, \quad y = -5 - t, \quad z = 8t$$

$$L_2: \frac{x+2}{6} = \frac{1-y}{2} = \frac{z+7}{16}$$

$$L_1: 3\hat{i} - \hat{j} + 8\hat{k}$$

$$L_2: 6\hat{i} - 2\hat{j} + 16\hat{k}$$

} DIRECTION VECTORS ARE SCALAR
MULTIPLES \Rightarrow VECTORS ARE
PARALLEL $\Rightarrow L_1 \parallel L_2$

3. (10 points) Find an equation of the plane passing through the points $P(0, 2, 1)$, $Q(2, -1, 3)$, and $R(5, 0, 8)$.

$$\vec{PQ} = 2\hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{PR} = 5\hat{i} - 2\hat{j} + 7\hat{k}$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 2 \\ 5 & -2 & 7 \end{vmatrix}$$

$$= \hat{i}(-21+4) - \hat{j}(14-10) + \hat{k}(-4+15)$$

$$= -17\hat{i} - 4\hat{j} + 11\hat{k}$$

$$-17x - 4y + 11z = 3$$

Using \vec{n} AND $P(0, 2, 1)$

$$\rightarrow \text{PLANE IS } -17(x-0) - 4(y-2) + 11(z-1) = 0$$

4. (10 points) Let $f(x, y, z) = x^2yz^3$. Find the gradient vector and the maximum directional derivative at the point $(1, 2, 1)$.

$$\vec{\nabla} f(x, y, z) = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

$$\boxed{\vec{\nabla} f(1, 2, 1) = 4\hat{i} + \hat{j} + 6\hat{k}}$$

MAX DIRECTIONAL DERIVATIVE IS IN DIRECTION

OF GRADIENT...

$$\frac{\vec{\nabla} f(1, 2, 1) \cdot \vec{\nabla} f(1, 2, 1)}{\|\vec{\nabla} f(1, 2, 1)\|} = \|\vec{\nabla} f(1, 2, 1)\|$$

$$= \sqrt{16 + 1 + 36}$$

$$= \boxed{\sqrt{53}}$$

5. (10 points) Find the critical points of $g(x,y) = x^3 + y^3 - 3xy + 9$. Then use the 2nd partials test to classify the critical points and determine the relative extrema.

$$\begin{aligned} g_x(x,y) &= 3x^2 - 3y = 0 \Rightarrow y = x^2 \\ g_y(x,y) &= 3y^2 - 3x = 0 \Rightarrow x = y^2 \end{aligned} \quad \left. \begin{array}{l} y = y^4 \\ x = y^2 \end{array} \right\} \quad y = y^4 \Rightarrow y = 0 \text{ or } y = 1$$

$\downarrow \quad \downarrow$
 $x = 0 \quad x = 1$

Two crit pts: $(0,0)$ & $(1,1)$

$$d(x,y) = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} \quad d(0,0) = -9 \Rightarrow (0,0,9) \text{ IS A SADDLE PT.}$$

$$d(1,1) = 27$$

$\& g_{xx}(1,1) = 6 > 0 \Rightarrow g(1,1) = 8 \text{ IS A REL MIN}$

6. (10 points) A baseball is hit from 2.8 ft above home plate with an initial velocity vector of $82\hat{i} + 78\hat{j}$ (measured in ft/s). Will the ball clear a 20-ft wall 380 ft from home plate. Neglect all forces except gravity and use $g = 32 \text{ ft/s}^2$.

$$\vec{r}(t) = 82t\hat{i} + (-16t^2 + 78t + 2.8)\hat{j}$$

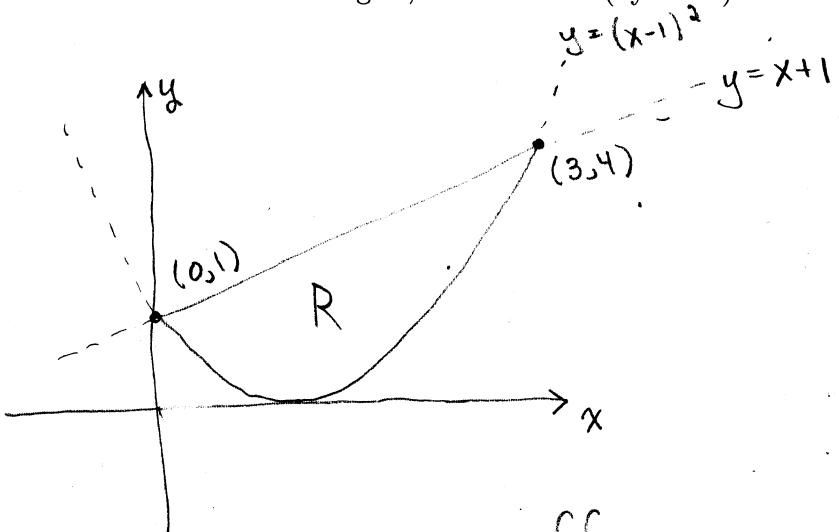
$$82t = 380 \Rightarrow t \approx 4.63415 \text{ sec}$$

$$-16t^2 + 78t + 2.8 \approx 20.658 \text{ FT}$$

$$\text{when } t \approx 4.63415 \text{ sec}$$

THE BALL WILL CLEAR THE
FENCE.

7. (12 points) A thin plate of constant density ρ lies in the 1st quadrant bounded by the graphs of $y = (x - 1)^2$ and $y = x + 1$. Write the double integral that gives the moment about the y -axis. Then sketch the region of integration, write the integral as an iterated integral, and evaluate (by hand).



$$\begin{aligned}
 M_y &= \iint_R x \rho dA \\
 &= \int_{x=0}^{x=3} \int_{y=(x-1)^2}^{y=x+1} \rho x dy dx \\
 &= \int_0^3 \rho x \left[(x+1) - (x-1)^2 \right] dx \\
 &= \int_0^3 \rho \left(-x^3 + 3x^2 \right) dx \\
 &= \rho \left(x^3 - \frac{1}{4}x^4 \right) \Big|_0^3 = \rho \left(27 - \frac{81}{4} \right)
 \end{aligned}$$

$$= \boxed{\frac{27\rho}{4}}$$

8. (12 points) Determine the limit or show that it does not exist.

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$$

CONVERT TO POLAR...

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{\sqrt{r^2}} &= \lim_{r \rightarrow 0} |r| (\cos^2 \theta - \sin^2 \theta) \\ &= \boxed{0} \end{aligned}$$

$$(b) \lim_{(x,y) \rightarrow (0,0)} \frac{3xy^2}{x^2 + y^4}$$

Along $x=0$: $\lim_{y \rightarrow 0} \frac{0}{y^4} = \lim_{y \rightarrow 0} 0 = 0$

Along $x=y^2$: $\lim_{y \rightarrow 0} \frac{3y^4}{2y^4} = \lim_{y \rightarrow 0} \frac{3}{2} = \frac{3}{2}$

LIMIT DNE

9. (10 points) The position of a particle at time t is given by $\vec{r}(t) = e^t \hat{i} + 2e^{-t} \hat{j} + t \hat{k}$. Write the definite integral that gives the length of the particle's path from $t = 0$ to $t = \ln 3$. Use your calculator to approximate the value of the integral.

$$\text{Arc Length} = \int_0^{\ln 3} \|\vec{v}(t)\| dt, \quad \vec{v}(t) = e^t \hat{i} - 2e^{-t} \hat{j} + \hat{k}$$

$$= \int_0^{\ln 3} \sqrt{e^{2t} + 4e^{-2t} + 1} dt \approx 2.73$$

10. (12 points) Consider the vector field

$$\vec{F}(x, y, z) = (y + y^2 z)\hat{i} + (x - z + 2xyz)\hat{j} + (-y + xy^2)\hat{k}.$$

(a) Find a scalar potential function $f(x, y, z)$ for $\vec{F}(x, y, z)$.

$$\frac{\partial f}{\partial x} = y + y^2 z \Rightarrow f(x, y, z) = xy + xy^2 z + g_1(y, z)$$

$$\frac{\partial f}{\partial y} = x - z + 2xyz \Rightarrow f(x, y, z) = xy - yz + xy^2 z + g_2(x, z)$$

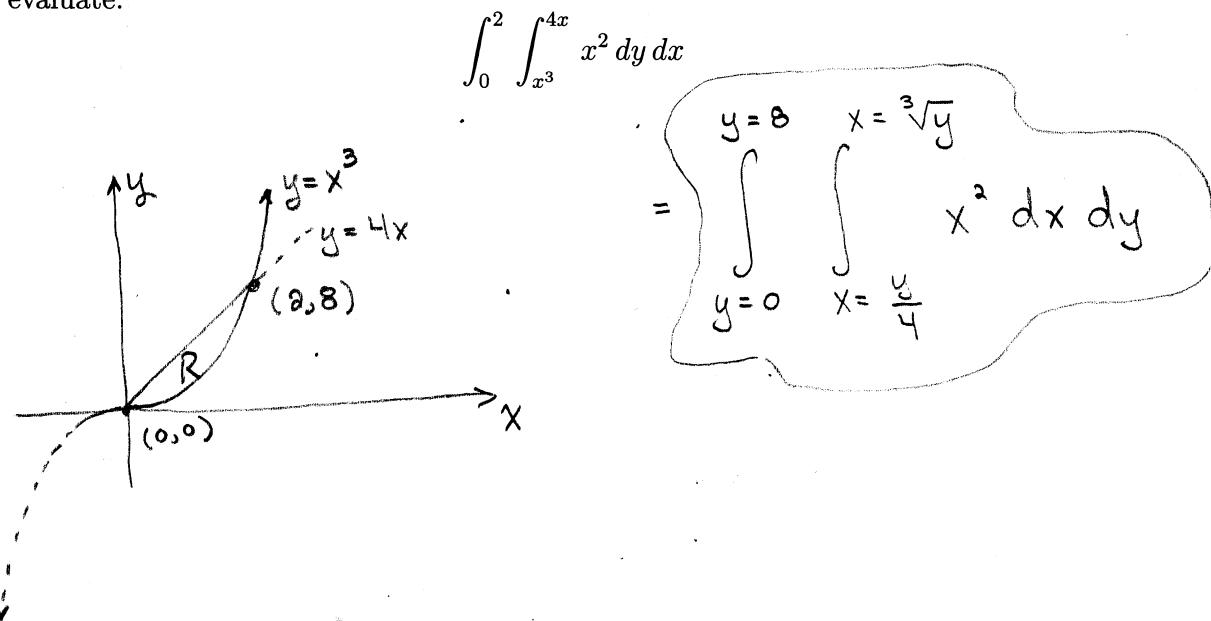
$$\frac{\partial f}{\partial z} = -y + xy^2 \Rightarrow f(x, y, z) = -yz + xy^2 z + g_3(x, y)$$

$$f(x, y, z) = xy - yz + xy^2 z$$

(b) Use any method to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the line segment joining $(2, 2, 1)$ and $(1, -1, 2)$.

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= f(1, -1, 2) - f(2, 2, 1) \\ &= [-1 + 2 + 2] - [4 - 2 + 8] \\ &= 3 - 10 = \boxed{-7} \end{aligned}$$

11. (8 points) Write the iterated integral with the reversed order of integration. Do not evaluate.



12. (10 points) Find an equation of the plane tangent to the graph of $f(x, y) = \frac{5}{x^2 + y^2}$ at the point where $(x, y) = (-1, 2)$.

$$f(-1, 2) = 1$$

$$F(x, y, z) = z - \frac{5}{x^2 + y^2}$$

Our surface is the level surface

$$F(x, y, z) = 0.$$

$$\vec{\nabla} F(x, y, z) = \left(\frac{10x}{(x^2 + y^2)^2} \hat{i} + \left(\frac{10y}{(x^2 + y^2)^2} \hat{j} + \hat{k} \right) \right)$$

$$\vec{n} = \vec{\nabla} F(-1, 2, 1)$$

$$= -\frac{10}{25} \hat{i} + \frac{20}{25} \hat{j} + \hat{k}$$

$$= -\frac{2}{5} \hat{i} + \frac{4}{5} \hat{j} + \hat{k}$$

Tangent Plane is

$$-\frac{2}{5}(x+1) + \frac{4}{5}(y-2) + (z-1) = 0$$

or

$$-\frac{2}{5}x + \frac{4}{5}y + z = 3$$

13. (12 points) Use the method of Lagrange multipliers to find the extreme values of $f(x, y) = 2x^2 + y^2 + 5$ subject to the constraint $\underbrace{x^2 + 4y^2 = 4}_{g(x,y)}$.

$$\vec{\nabla} f(x,y) = 4x \hat{i} + 2y \hat{j}$$

$$\vec{\nabla} g(x,y) = 2x \hat{i} + 8y \hat{j}$$

$$4x = \lambda 2x \quad \lambda 2x - 4x = 0 \Rightarrow 2x(\lambda - 2) = 0 \Rightarrow x = 0 \text{ or } \lambda = 2$$

$$2y = \lambda 8y \quad \lambda 8y - 2y = 0 \Rightarrow 2y(4\lambda - 1) = 0 \quad \begin{matrix} \downarrow \\ y = \pm 1 \end{matrix} \quad \begin{matrix} \downarrow \\ y = 0 \end{matrix}$$

$$x^2 + 4y^2 = 4$$

$$f(0, \pm 1) = 6 \leftarrow \text{Abs min}$$

$$f(\pm 2, 0) = 13 \leftarrow \text{Abs max}$$

$$x = \pm 2 \quad x = 0$$

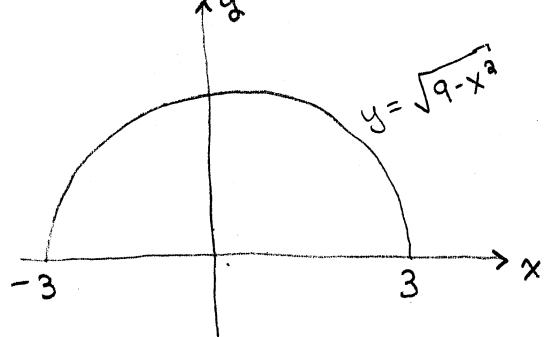
Crit pts are

$$(0, 1), (0, -1),$$

$$(2, 0), (-2, 0)$$

14. (12 points) Convert to cylindrical coordinates and evaluate.

$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \int_0^2 \frac{1}{1+x^2+y^2} dz dy dx$$



$$\int_{\theta=0}^{\pi} \int_{r=0}^3 \int_{z=0}^2 \frac{r}{1+r^2} dz dr d\theta$$

$$= 2\pi \int_0^3 \frac{r}{1+r^2} dr \quad u = 1+r^2 \quad du = 2r dr$$

$$= \pi \int_1^{10} \frac{1}{u} du = \boxed{\pi \ln 10}$$