Math 173 - Test 1

February 23, 2017

Name key

Score ________

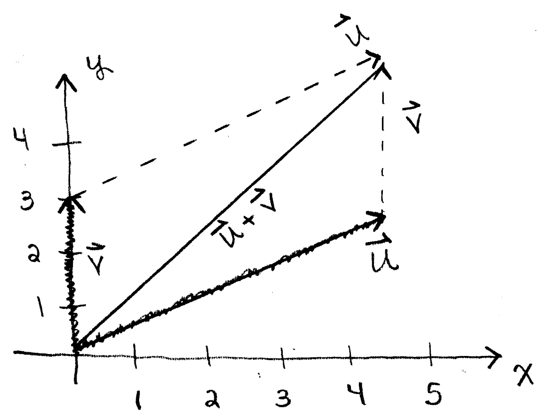
Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) The vector $\vec{u}$ is the 2D vector that has magnitude 5 and makes a 30° angle with the positive x-axis. The vector $\vec{v}$ is the 2D vector that has magnitude 3 and makes a 90° angle with the positive x-axis. Compute $\vec{u}+\vec{v}$. Then sketch $\vec{u}$, $\vec{v}$, and $\vec{u}+\vec{v}$ in the xy-plane and indicate how these vectors are related by the parallelogram law.

$$\vec{u} = 5\cos 30^\circ \hat{i} + 5\sin 30^\circ \hat{j} = \frac{5\sqrt{3}}{2}\hat{i} + \frac{5}{2}\hat{j}$$

$$\vec{v} = 3\hat{j}$$

$$\boxed{\vec{u}+\vec{v} = \frac{5\sqrt{3}}{2}\hat{i} + \frac{11}{2}\hat{j}}$$



$\vec{u}+\vec{v}$ = DIAG OF PARALLELOGRAM

2. (6 points) Use vectors to show that the points A, B, and C are collinear.

$A\,(-7, 9, 10) \qquad B\,(3, 4, -5) \qquad C\,(1, 5, -2)$

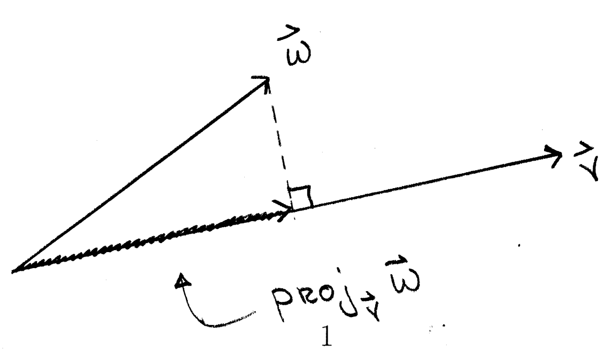
$$\vec{AB} = 10\hat{i} - 5\hat{j} - 15\hat{k}$$

$$\vec{AC} = 8\hat{i} - 4\hat{j} - 12\hat{k}$$

$$\vec{AC} = \frac{4}{5}\vec{AB}$$

$\vec{AC}$ IS A SCALAR MULTIPLE OF $\vec{AB}$

3. (4 points) Sketch a diagram that shows two vectors, $\vec{v}$ and $\vec{w}$, and then show the vector $\text{proj}_{\vec{v}}\,\vec{w}$.



4. (4 points) What does it mean for two vectors to be orthogonal? Use your answer to show that $\vec{u} = 3\hat{\imath} - 7\hat{\jmath} + \hat{k}$ is orthogonal to $\vec{v} = -5\hat{\imath} - 2\hat{\jmath} + \hat{k}$.

TWO VECTORS ARE ORTHOGONAL IF AND ONLY IF THEIR DOT PRODUCT IS ZERO.

$$\vec{u} \cdot \vec{v} = 3(-5) + (-7)(-2) + (1)(1)$$
$$= -15 + 14 + 1 = 0$$

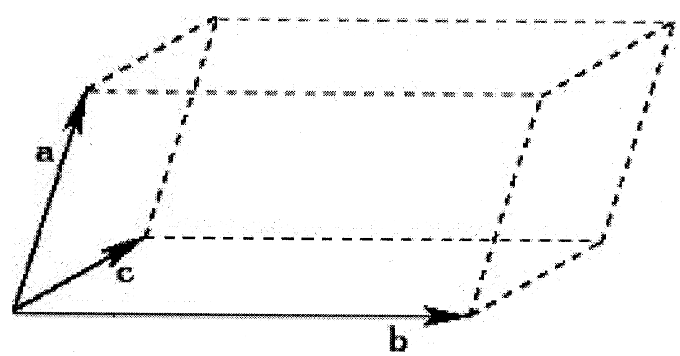
5. (4 points) If the projection of $\vec{u}$ onto $\vec{v}$ has the same magnitude as the projection of $\vec{v}$ onto $\vec{u}$, can you conclude that $\|\vec{u}\| = \|\vec{v}\|$? Explain.

$$\|\mathrm{proj}_{\vec{v}}\, \vec{u}\| = \|\mathrm{proj}_{\vec{u}}\, \vec{v}\| \Rightarrow \frac{|\vec{u}\cdot\vec{v}|}{\vec{v}\cdot\vec{v}}\|\vec{v}\| = \frac{|\vec{u}\cdot\vec{v}|}{\vec{u}\cdot\vec{u}}\|\vec{u}\|$$

$$\Rightarrow \frac{1}{\|\vec{v}\|} = \frac{1}{\|\vec{u}\|} \Rightarrow \|\vec{v}\| = \|\vec{u}\|$$

So, Yes!

6. (8 points) A crystal structure has the form of a parallelepiped determined by the vectors $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = -\hat{\imath} + \hat{\jmath} - 4\hat{k}$, and $\vec{c} = 4\hat{\imath} + 2\hat{\jmath} - \hat{k}$, where distances are measured in micrometers. Find the volume of the parallelepiped.



$$\text{VOLUME} = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 3 & -2 & 1 \\ -1 & 1 & -4 \\ 4 & 2 & -1 \end{vmatrix} = 3(-1+8) - (-2)(1+16) + (1)(-2-4)$$
$$= 21 + 34 - 6 = 49$$

VOLUME = 49 μm^3

7. (12 points) The points $P(0,2,3)$, $Q(-1,2,4)$, and $R(3,-7,2)$ are the vertices of a triangle.

(a) Find the area of the triangle.

$\vec{PQ} = -\hat{i} + \hat{k}$

$\vec{PR} = 3\hat{i} - 9\hat{j} - \hat{k}$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ 3 & -9 & -1 \end{vmatrix}$$

$$= \hat{i}(0+9) - \hat{j}(1-3) + \hat{k}(9-0) = 9\hat{i} + 2\hat{j} + 9\hat{k}$$

$$\text{AREA} = \frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$$
$$= \frac{1}{2}\sqrt{81+4+81}$$
$$= \frac{\sqrt{166}}{2} \approx 6.442$$

(b) Find an equation of the plane containing the triangle.

$$9x + 2y + 9z = 9(0) + 2(2) + 9(3) = 31$$

$$\boxed{9x + 2y + 9z = 31}$$

(c) Find a set of parametric equations for the line segment $\overline{PR}$.

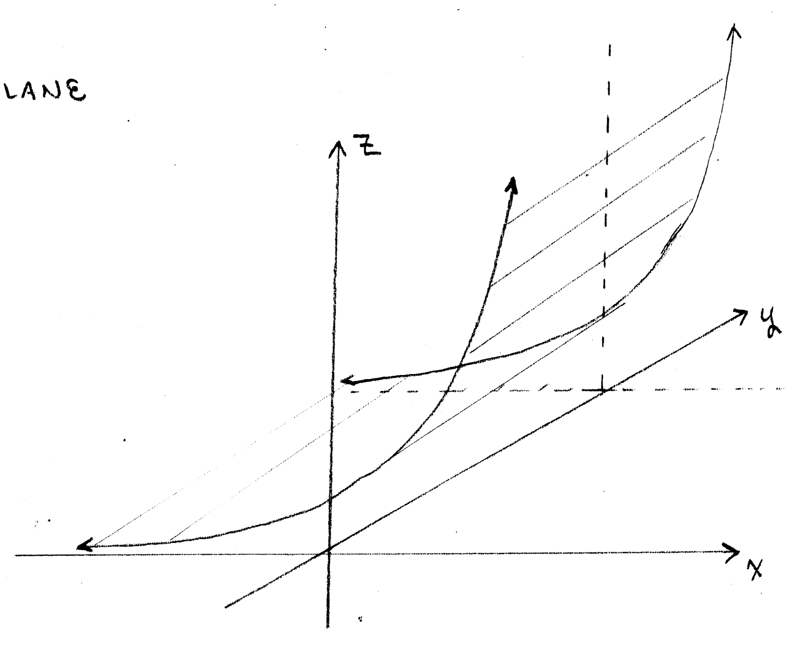
$\vec{PR} = 3\hat{i} - 9\hat{j} - \hat{k}$

$P\ (0,2,3)$

$$x = 3t$$
$$y = 2 - 9t \qquad 0 \le t \le 1$$
$$z = 3 - t$$

8. (5 points) Describe (or sketch) the 3D surface defined by the equation $z = e^x$.

$z = e^x$ DESCRIBES THE CYLINDER WHOSE GENERATING CURVE IS THE CURVE $z = e^x$ IN THE XZ-PLANE AND WHOSE REFERENCE LINE IS THE y-AXIS.



9. (7 points) Find the vector-valued function $\vec{r}(t)$ such that

$$\vec{r}\,'(t) = te^{-t^2}\hat{\imath} - e^{-t}\hat{\jmath} + \hat{k}; \quad \vec{r}(0) = \frac{1}{2}\hat{\imath} - \hat{\jmath} + 2\hat{k}.$$

$$\int t e^{-t^2} dt$$

$u = -t^2$

$du = -2t\,dt$

$-\frac{1}{2} du = t\,dt$

$$-\frac{1}{2}\int e^u du$$

$$= -\frac{1}{2}e^u + C$$

$$= -\frac{1}{2}e^{-t^2} + C$$

$$\vec{r}(t) = \left(-\frac{1}{2}e^{-t^2} + C_1\right)\hat{\imath} + \left(e^{-t} + C_2\right)\hat{\jmath} + \left(t + C_3\right)\hat{k}$$

$$\vec{r}(0) = \left(-\frac{1}{2} + C_1\right)\hat{\imath} + \left(1 + C_2\right)\hat{\jmath} + C_3\hat{k}$$

$$= \frac{1}{2}\hat{\imath} - \hat{\jmath} + 2\hat{k}$$

$\Rightarrow C_1 = 1,\ C_2 = -2,\ C_3 = 2$

$$\vec{r}(t) = \left(-\frac{1}{2}e^{-t^2} + 1\right)\hat{\imath} + \left(e^{-t} - 2\right)\hat{\jmath} + \left(t + 2\right)\hat{k}$$

10. (10 points) A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 meters. Find the required initial velocity and the maximum height of the projectile.

$$\vec{r}(t) = V_0 \cos 8^\circ t\,\hat{\imath} + \left(-4.9t^2 + V_0 \sin 8^\circ t\right)\hat{\jmath}$$

$$V_0 \cos 8^\circ t = 50 \Rightarrow V_0 t = \frac{50}{\cos 8^\circ}$$

$$-4.9t^2 + V_0 \sin 8^\circ t = 0$$

$$-4.9t^2 + \sin 8^\circ\left(\frac{50}{\cos 8^\circ}\right) = 0$$

$$t = \sqrt{\frac{50 \tan 8^\circ}{4.9}} \approx 1.1975 \text{ sec} \Rightarrow V_0 = \frac{\frac{50}{\cos 8^\circ}}{t} \approx 42.16 \text{ m/sec}$$

SINCE STARTS AND ENDS ON GROUND, MAX HEIGHT AT MIDDLE OF FLIGHT.

$$t \approx \frac{1.1975}{2} \text{ SEC}$$

$$-4.9t^2 + V_0 \sin 8^\circ t$$

$$\approx 1.76\text{ m}$$

11. (1 point) If you were given two nonparallel vectors, how could you find a nonzero vector orthogonal to both?

CROSS PRODUCT

IS ORTHOG. TO BOTH.

12. (4 points) What is the domain of the function $\vec{r}(t) = \sqrt{t}\,\hat{\imath} + \tan t\,\hat{\jmath} + \ln(t-1)\,\hat{k}$?

$t \geq 0$ AND $t \neq$ ODD MULTIPLE OF $\frac{\pi}{2}$. AND $t > 1$

$t > 1$ AND. $t \neq$ ODD MULTIPLE OF $\frac{\pi}{2}$

13. (3 points) Refer to the function above. Is $\vec{r}(t)$ continuous at $t = 4$? Explain.

YES $\vec{r}$ IS CONTINUOUS WHEREVER IT IS DEFINED, AND IT IS DEFINED AT $t=4$.

$\lim_{t \to 4} \vec{r}(t) = \vec{r}(4)$

14. (3 points) Suppose $\vec{r}(t)$ describes a line in space. What can be said about $\hat{T}'(t)$? Explain.

$\vec{r}(t) = (at + x_0)\hat{\imath} + (bt + y_0)\hat{\jmath} + (ct + z_0)\hat{k}$

$\Rightarrow \hat{T}(t) =$ CONSTANT VECTOR

$\Rightarrow \hat{T}'(t) = \vec{0}$.

15. (6 points) Let $\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + \ln(t)\hat{k}$. Compute $\hat{T}(1)$.

$\vec{r}'(t) = \hat{\imath} + 2t\hat{\jmath} + \frac{1}{t}\hat{k}$.

$\vec{r}'(1) = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

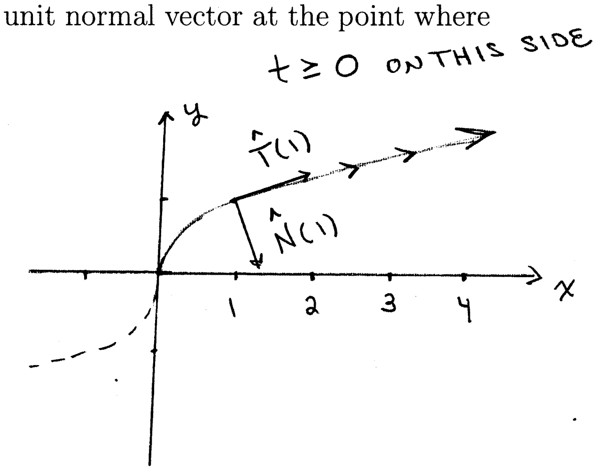
$\hat{T}(1) = \frac{\vec{r}'(1)}{\|\vec{r}'(1)\|} = \frac{\hat{\imath} + 2\hat{\jmath} + \hat{k}}{\sqrt{6}}$

16. (8 points) For $t \geq 0$, sketch the graph of $\vec{r}(t) = t^3\hat{\imath} + t\hat{\jmath}$. Without computing them, sketch the unit tangent vector and the principal unit normal vector at the point where $t = 1$.

$$x = t^3, \quad y = t \Rightarrow x = y^3$$

OR

$$y = \sqrt[3]{x}$$



17. (3 points) The angle between $\vec{u}$ and $\vec{v}$ is obtuse. What can be said about $\vec{u} \cdot \vec{v}$? Briefly explain.

SINCE $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

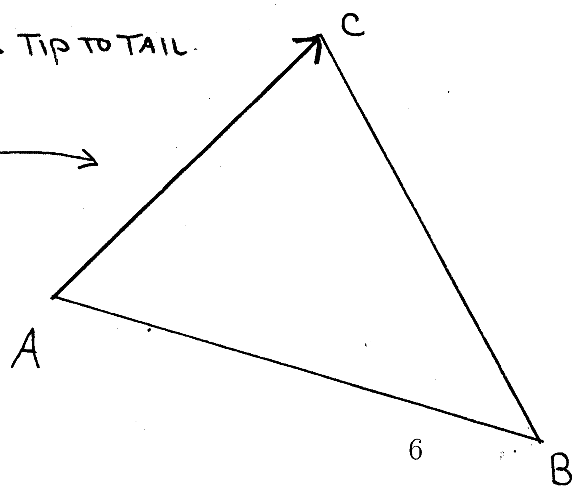
AND $\cos\theta < 0$ FOR OBTUSE $\angle$'s,

$\vec{u} \cdot \vec{v}$ IS NEGATIVE.

18. (4 points) Let A, B, and C be the vertices of a triangle. Determine $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

VECTOR ADDITION IS TIP TO TAIL.

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

AND

$\overrightarrow{AC} = -\overrightarrow{CA}$

So,

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$