# Math 173 - Test 3a
April 27, 2017

Name key

Score

Show all work. Supply explanations when necessary.

1. (4 points) Suppose $y$ is implicitly defined as a function of $x$ by the equation

$$\ln\sqrt{x^2+y^2}+x+xy=4.$$

Find $dy/dx$.

$$\underbrace{\frac{1}{2}\ln(x^2+y^2)+x+xy}_{F(x,y)}=4$$

$$\frac{dy}{dx}=\frac{-F_x}{F_y}=\frac{-\left(\frac{x}{x^2+y^2}+1+y\right)}{\frac{y}{x^2+y^2}+x}$$

2. (5 points) Find the directional derivative of $f(x,y)=e^y\sin x$ at the point $(0,0)$ in the direction of $(2,1)$.

$$\vec{\nabla}f(x,y)=e^y\cos x\,\hat{i}+e^y\sin x\,\hat{j}$$

$$\vec{\nabla}f(0,0)=\hat{i}$$

$$\vec{u}=(2-0)\hat{i}+(1-0)\hat{j}=2\hat{i}+\hat{j}$$

$$\|\vec{u}\|=\sqrt{5}$$

$$D_{\vec{u}}f(0,0)=\frac{1}{\|\vec{u}\|}\vec{\nabla}f(0,0)\cdot\vec{u}=\frac{2}{\sqrt{5}}$$

3. (5 points) In this problem you will find the extreme values of $f(x, y, z) = x^2 - y + yz$ subject to the constraint $x + y = z^2$.

(a) Set up, but do not solve, the system of equations that is obtained by applying the Lagrange multiplier method to this problem.

$f(x,y,z) = x^2 - y + yz$

$g(x,y,z) = x + y - z^2$

$\vec{\nabla} f(x,y,z) = 2x\,\hat{i} + (z-1)\,\hat{j} + y\,\hat{k}$

$\vec{\nabla} g(x,y,z) = \hat{i} + \hat{j} - 2z\,\hat{k}$

$$\boxed{\begin{aligned} 2x &= \lambda \\ z-1 &= \lambda \\ y &= -2\lambda z \\ x+y &= z^2 \end{aligned}}$$

(b) Your system of equations has two solutions for $(x, y, z)$. They are

$$\left(-\frac{1}{3}, \frac{4}{9}, \frac{1}{3}\right) \quad \text{and} \quad \left(-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right)$$

Use this information to find the maximum and minimum values of $f(x, y, z)$ on the constraint surface.

$f\left(-\frac{1}{3}, \frac{4}{9}, \frac{1}{3}\right) = -\frac{5}{27} \approx -0.1852 \leftarrow$ MAX VALUE

$f\left(-\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right) = -\frac{3}{16} = -0.1875 \leftarrow$ MIN VALUE

4. (5 points) Find and classify the critical points of $f(x, y) = x^2 - xy - y^2 - 3x - y$.

$f_x(x,y) = 2x - y - 3 = 0$

$f_y(x,y) = -x - 2y - 1 = 0$

$$D(x,y) = \begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = -4 - 1 = -5$$

$$\begin{aligned} x + 2y &= -1 \\ 2(2x - y &= 3) \\ \hline 5x &= 5 \end{aligned}$$

$x = 1$

$2(1) - y = 3$

$\Rightarrow y = -1$

$(1,-1)$

$$\boxed{\begin{aligned} &(1, -1, f(1,-1)) \\ &= (1, -1, -1) \\ &\text{IS A SADDLE PT.} \end{aligned}}$$

5. (5 points) Find an equation of the plane tangent to the surface $\underbrace{xy^2 + 3x - z^2}_{f(x,y,z)} = 8$ at the point $(1, -3, 2)$.

$f(1,-3,2) = 9 + 3 - 4 = 8$ ✓

$\vec{\nabla} f(1,-3,2)$ IS NORMAL TO THE SURFACE AT $(1,-3,2)$

$\vec{\nabla} f(x,y,z) = (y^2 + 3)\hat{i} + 2xy\hat{j} - 2z\hat{k}$

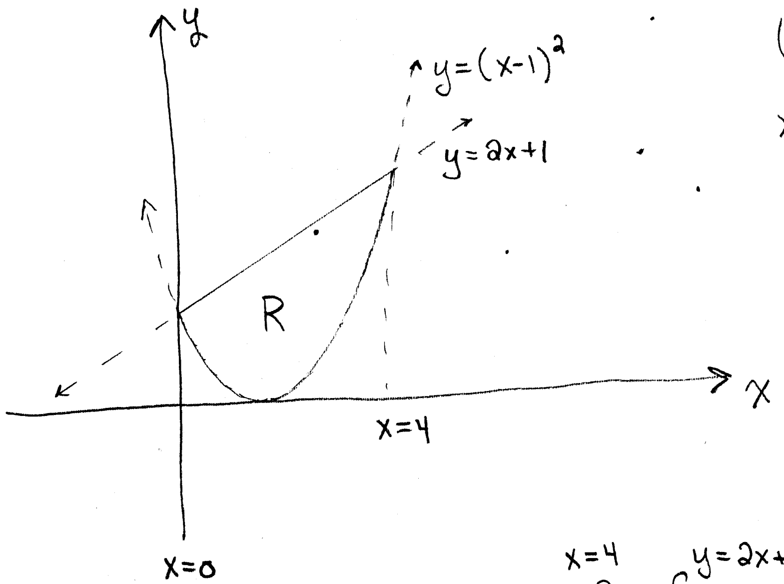
$\vec{\nabla} f(1,-3,2) = 12\hat{i} - 6\hat{j} - 4\hat{k}$

TAN PLANE IS $12(x-1) + (-6)(y+3) - 4(z-2) = 0$

OR

$$\boxed{12x - 6y - 4z = 22}$$

6. (7 points) Evaluate the double integral $\iint_R x^2\,dA$, where $R$ is the first quadrant region bounded by the graphs of $y = (x-1)^2$ and $y = 2x+1$.




$$(x-1)^2 = 2x+1$$

$$x^2 - 2x + 1 = 2x + 1$$

$$x^2 - 4x = 0$$

$$x = 0,\ x = 4$$

$$\int_{x=0}^{x=4} \int_{y=(x-1)^2}^{y=2x+1} x^2 \, dy \, dx$$

$$= \int_0^4 x^2 y \Big|_{(x-1)^2}^{2x+1} dx = \int_0^4 x^2\left[2x+1-(x-1)^2\right] dx$$

$$= \int_0^4 \left(-x^4 + 4x^3\right) dx = -\frac{1}{5}x^5 + x^4 \Big|_0^4$$

$$= -\frac{4^5}{5} + 4^4 = \boxed{\frac{256}{5}}$$

**Math 173 - Test 3b**
April 27, 2017

Name key

Score

Show all work. Supply explanations when necessary. You must work individually on this test.

1. (6 points) Suppose $z$ is implicitly defined as a function of $x$ and $y$ by the equation

$$\underbrace{x \ln y + y^2 z + z^2}_{F(x,y,z)} = 8.$$

Find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-\ln y}{y^2 + 2z}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\left(\frac{x}{y} + 2yz\right)}{y^2 + 2z}$$

2. (6 points) Find the point(s) on the surface described by $z = 3x^2 + 2y^2 - 3x + 4y - 5$ at which the tangent plane is horizontal.

TAN PLANE IS HORIZONTAL WHERE GRADIENT IS $\vec{0}$.

$$\vec{\nabla} z = (6x-3)\hat{i} + (4y+4)\hat{j} = \vec{0}$$

$$\Rightarrow \quad x = \frac{1}{2},\ y = -1 \Rightarrow z = \frac{3}{4} + 2 - \frac{3}{2} - 4 - 5 = \frac{-31}{4}$$

$$\Rightarrow \boxed{(x,y,z) = \left(\frac{1}{2}, -1, -\frac{31}{4}\right)}$$

3. (6 points) The temperature at the point $(x, y)$ on a metal plate is given by

$$T(x, y) = 4 + \sin(xy) + x + xy, \quad -2 \le x \le 2,\ -2 \le y \le 2.$$

(a) Find the direction of greatest increase in temperature from the point $(0, 1)$.

$$\vec{\nabla} T(x,y) = (y \cos xy + 1 + y)\,\hat{i} + (x \cos xy + x)\,\hat{j}$$

$$\vec{\nabla} T(0,1) = 3\,\hat{i}$$

$\boxed{\text{DIRECTION OF } \hat{i}}$

(b) At which point is there no increase or decrease in temperature regardless of which direction we look?

$$\vec{\nabla} T(x,y) = \vec{0} \Rightarrow \quad y \cos xy + 1 + y = 0$$

$$\underbrace{x \cos xy + x = 0}$$

$x = 0$ OR $\cos xy = -1$

$y + 1 + y = 0 \rightarrow y = -\frac{1}{2}$

$-y + 1 + y = 0 \rightarrow$ No WAY.

$$\boxed{\left(0, -\tfrac{1}{2}\right)}$$

4. (10 points) Find and classify the critical points of $g(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$. Find all relative extreme values.

$g_x(x,y) = -6xy - 6x = 0 \Rightarrow -6x(y+1) = 0$

$g_y(x,y) = 3y^2 - 3x^2 - 6y = 0$

$x = 0 \quad\quad y = -1$

$3y^2 - 6y = 0 \quad\quad 3 - 3x^2 + 6 = 0$

$3y(y-2) = 0 \quad\quad 3x^2 = 9$

$x^2 = 3$

$y = 0 \quad y = 2 \quad\quad x = \pm\sqrt{3}$

$(0,0),\ (0,2),\ (\sqrt{3},-1),\ (-\sqrt{3},-1)$

$$D(x,y) = \begin{vmatrix} -6y-6 & -6x \\ -6x & 6y-6 \end{vmatrix} = -36(y^2-1) - 36x^2$$
$$= -36\left[(y^2-1) + x^2\right]$$

$D(0,0) = 36$ & $g_{xx}(0,0) = -6$

$\Rightarrow$ $g(0,0) = 1$ IS A REL MAX

$D(0,2) = -108 \Rightarrow (0, 2, g(0,2)) = (0,2,-3)$ IS A SADDLE PT

$D(\sqrt{3},-1) = -108 \Rightarrow (\sqrt{3}, -1, g(\sqrt{3},-1)) = (\sqrt{3},-1,-3)$ IS A SADDLE PT

$D(-\sqrt{3},-1) = -108 \Rightarrow (-\sqrt{3},-1, g(-\sqrt{3},-1)) = (-\sqrt{3},-1,-3)$ IS A SADDLE PT

5. (8 points) Find the minimum and maximum values of $g(x,y) = e^{-xy/4}$ on the circle $\underbrace{x^2 + y^2}_{h(x,y)} = 1$.

$$\vec{\nabla} g(x,y) = -\frac{y}{4} e^{-xy/4}\,\hat{\imath} - \frac{x}{4} e^{-xy/4}\,\hat{\jmath}$$

$$\vec{\nabla} h(x,y) = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$$

$$\left.\begin{aligned} -\frac{y}{4} e^{-xy/4} &= \lambda\, 2x \\ -\frac{x}{4} e^{-xy/4} &= \lambda\, 2y \end{aligned}\right\}$$

$$x^2 + y^2 = 1$$

NOTICE THAT WE CANNOT HAVE $X = 0$ OR $y = 0$

$$\frac{y}{x} = \frac{x}{y} \quad \text{OR} \quad x^2 = y^2.$$

$$2x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

CANDIDATES ARE

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

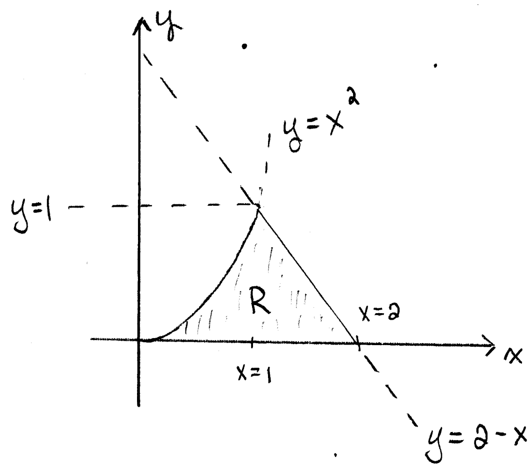
$$g\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = e^{-1/8} \leftarrow \text{MIN VALUE}$$

$$g\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = e^{1/8} \leftarrow \text{MAX VALUE}$$

6. (15 points) Suppose that

$$\iint_R 4x\,dA = \int_0^1 \int_0^{x^2} 4x\,dy\,dx + \int_1^2 \int_0^{2-x} 4x\,dy\,dx.$$

Sketch the region of integration $R$, reverse the order of integration, and evaluate the new iterated integral by hand.




$$\int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=y-2} 4x\,dx\,dy$$

$$= \int_0^1 2x^2 \Big|_{\sqrt{y}}^{2-y} dy$$

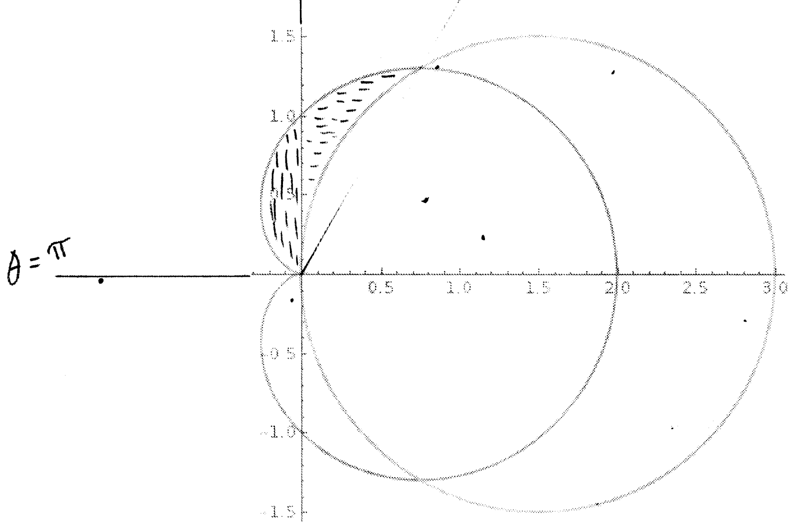
$$= \int_0^1 2(2-y)^2 - 2y\,dy$$

$$= \int_0^1 (8 - 10y + 2y^2)\,dy$$

$$= 8 - 5 + \frac{2}{3} = \boxed{\frac{11}{3}}$$

$$1+\cos\theta = 3\cos\theta \;\Rightarrow\; 1 = 2\cos\theta \;\Rightarrow\; \theta = \pi/3$$

7. (8 points) Use a double integral to find the area of the polar region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$. (Evaluate your integral by hand.)




$$2\cdot\int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=3\cos\theta}^{r=1+\cos\theta} r\,dr\,d\theta + 2\cdot\int_{\theta=\pi/2}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} r\,dr\,d\theta$$

$$= \int_{\pi/3}^{\pi/2} (1+\cos\theta)^2 - (3\cos\theta)^2\,d\theta + \int_{\pi/2}^{\pi} (1+\cos\theta)^2\,d\theta$$

$$= \int_{\pi/3}^{\pi/2} 1 + 2\cos\theta - 8\cos^2\theta\,d\theta + \int_{\pi/2}^{\pi} 1 + 2\cos\theta + \cos^2\theta$$

$$= \int_{\pi/3}^{\pi/2} 1 + 2\cos\theta - 4 - 4\cos 2\theta\,d\theta + \int_{\pi/2}^{\pi} 1 + 2\cos\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\,d\theta$$

$$= -3\theta + 2\sin\theta - 2\sin 2\theta \Big|_{\pi/3}^{\pi/2} + \tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta \Big|_{\pi/2}^{\pi}.$$

$$= \cancel{-\frac{3\pi}{2}} + 2 - 0 + \pi - \sqrt{3} + \sqrt{3} + \cancel{\frac{3}{2}\pi} + 0 + 0 - \frac{3\pi}{4} \cancel{- 2} - 0$$

$$= \boxed{\frac{\pi}{4}}$$

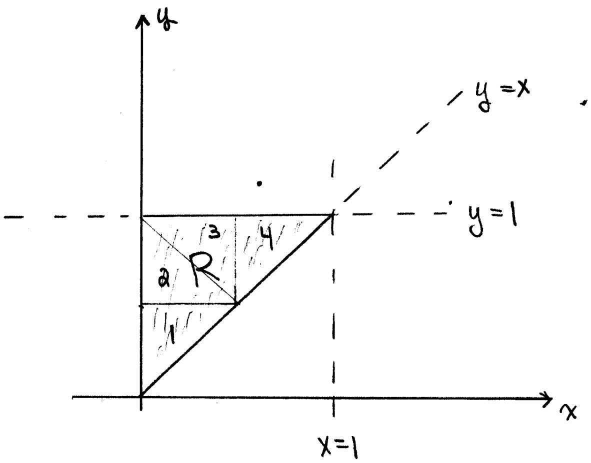
- OR -

COULD HAVE DONE

$$2\int_{r=0}^{r=3/2}\int_{\theta=\cos^{-1}\frac{r}{3}}^{\theta=\cos^{-1}(r-1)} r\,d\theta\,dr = \cdots = \frac{\pi}{4}$$

8. (10 points) Let $R$ be the 1st quadrant region bounded by the graphs of $y = x$, $y = 1$, and $x = 0$, and let $f(x, y) = 1 - xy$.

(a) Use a Riemann sum over 4 subregions to estimate the volume of the space region under the graph of $f$ and above $R$.




FOR EACH $i$, $\Delta A_i = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$

I'LL USE $(x_1, y_1) = \left(\frac{1}{4}, \frac{3}{8}\right)$

$(x_2, y_2) = \left(\frac{1}{4}, \frac{5}{8}\right)$

$(x_3, y_3) = \left(\frac{1}{4}, \frac{7}{8}\right)$

$(x_4, y_4) = \left(\frac{3}{4}, \frac{7}{8}\right)$

$$\sum_{i=1}^{4} f(x_i, y_i)\Delta A_i = \frac{1}{8}\left[\left(1 - \frac{3}{32}\right) + \left(1 - \frac{5}{32}\right) + \left(1 - \frac{7}{32}\right) + \left(1 - \frac{21}{32}\right)\right]$$

$$= \frac{1}{8}\left(4 - \frac{36}{32}\right) = \frac{23}{64} = 0.359375$$

(b) Find the volume of the space region by setting up and evaluating (by hand) an iterated integral.

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} (1 - xy)\, dy\, dx$$

$$= \int_0^1 \left(y - \frac{xy^2}{2}\right)\Bigg|_{y=x}^{y=1} dx = \int_0^1 \left(1 - \frac{x}{2} - x + \frac{x^3}{2}\right) dx$$

$$= 1 - \frac{1}{4} - \frac{1}{2} + \frac{1}{8} = \frac{3}{8} = 0.375$$